HW 11.9

① cylindrical to spherical $(x^2+y^2=r^2)$

$\rho = \sqrt{r^2+z^2}$

$\cos\phi = \dfrac{z}{\sqrt{r^2+z^2}}$

$\theta = \theta$

s to c

$r = \rho\sin\phi$

$z = \rho\cos\phi$

$\theta = \theta$

④ a) $x = 8\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$y = 8\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$z = 8\cos\left(\frac{\pi}{6}\right) = 4\sqrt{3}$

b) $(\sqrt{2}, \sqrt{6}, -2\sqrt{2})$

⑳ $x^2+y^2+4z^2=10$ spherical

$\left.\begin{array}{l} \underbrace{x^2+y^2+z^2}_{\rho^2} + 3z^2 = 10 \\ \rho^2 + 3\rho^2\cos^2\phi = 10 \end{array}\right\} \quad \rho^2 = \dfrac{10}{1+3\cos^2\phi}$

㉕ $x+y=4$ to cylindrical:

$\left.\begin{array}{l} r\cos\theta + r\sin\theta = 4 \end{array}\right\} \quad r = \dfrac{4}{\cos\theta+\sin\theta}$

㉚ $\rho\sin\phi = 1$ to cartesian

$\underbrace{\rho\sin\phi = 1}_{\text{spherical}} \Rightarrow r=1$ (cylindrical) $= \sqrt{x^2+y^2} = 1$ (cartesian)

$\Rightarrow x^2+y^2 = 1$

HW 12.1

(2) Let $f(x,y) = \frac{y}{x} + xy$

a) $f(1,2) = 2 + 2 = 4$

b) $f(\frac{1}{4}, 4) = \frac{4}{\frac{1}{4}} + 1 = 16 + 1 = 17$

c) $f(4, \frac{1}{4}) = \frac{1}{4 \cdot 4} + 1 = \frac{17}{16}$

d) $f(a,a) = 1 + a^2, \quad a \neq 0$

e) $f(\frac{1}{x}, x^2) = \frac{x^2}{\frac{1}{x}} + \frac{1}{x} \cdot x^2 = x^3 + x \quad x \neq 0$

f) $f(0,0) =$ undefined for $x = 0$.

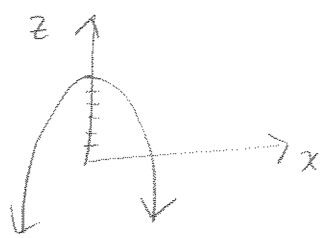
domain is all $x + y \ni x \neq 0$

(5) Find $F(f(t), g(t))$ if $F(x,y) = x^2 y$, $f(t) = t\cos t$, $g(t) = \sec^2 t$

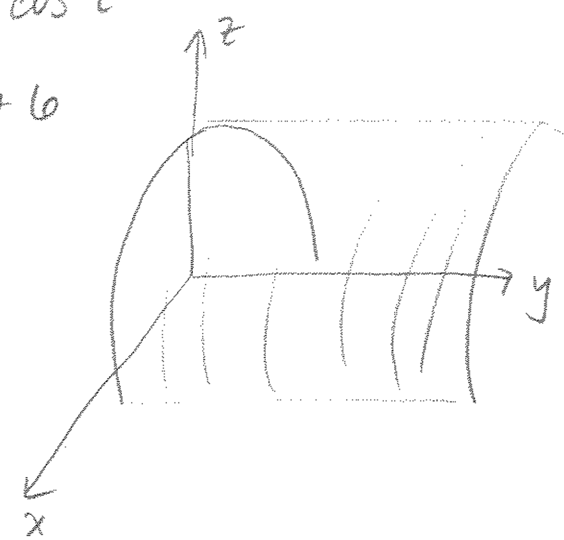
$F(f(t), g(t)) = (t\cos t)^2 (\sec^2 t) = t^2 \frac{\cos^2 t}{\cos^2 t} = t^2$ for $\cos^2 t \neq 0$

(10) sketch $f(x,y) = 6 - x^2 \Rightarrow z = -x^2 + 6$

in x-z plane:



$\Rightarrow$



parabolic cylinder.

(22) sketch level curves for $z = k$

$z = y - \sin x$, $k = -2, -1, 0, 1, 2$

$-2 = y - \sin x$

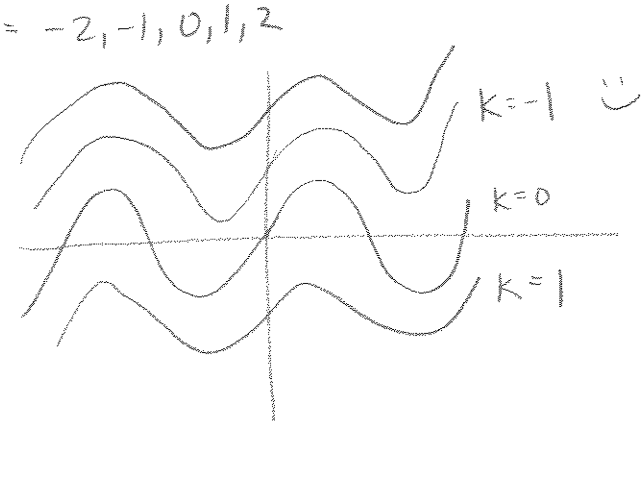
$k = -2 \quad y = 2 + \sin x$

$k = -1 \quad y = 1 + \sin x$

$k = 0 \quad y = \sin x$

$k = 1 \quad y = \sin x - 1$

$k = 2 \quad y = \sin x - 2$



12.2 HW

(6) $f(x,y) = (3x^2+y^2)^{-\frac{1}{3}}$

$f_x = -\frac{1}{3}(3x^2+y^2)^{-4/3}\,(6x)$

$f_y = -\frac{1}{3}(3x^2+y^2)^{-4/3}\,(2y)$

(16) $f(r,\theta) = 3r^3\cos 2\theta$

$f_r = 9r^2\cos 2\theta$

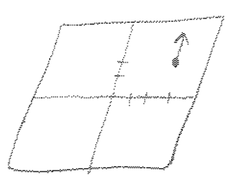
$f_\theta = 3r^3(-2\sin 2\theta) = -6r^3\sin 2\theta$

(19) $f(x,y) = 3e^{2x}\cos y$

$f_x = 6e^{2x}\cos y$ $\qquad$ $f_y = -3e^{2x}\sin y$

$f_{xy} = -6e^{2x}\sin y \;=\; f_{yx} = -6e^{2x}\sin y$ ✓

(30) $T(x,y) = 4 + 2x^2 + y^3$



holding x fixed at $x=2$

find $\frac{\partial T}{\partial y}$: $\quad \frac{\partial T}{\partial y} = 3y^2 \quad \frac{\partial T}{\partial y}(3,2) = 12$ degrees/ft

(33) show $f(x,y) = x^3y - xy^3$ is harmonic:

want $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 3x^2y - y^3 \qquad \frac{\partial f}{\partial y} = x^3 - 3xy^2$

$\frac{\partial^2 f}{\partial x^2} = 6xy \qquad \frac{\partial^2 f}{\partial y^2} = -6xy$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \cancel{6xy}\ 6xy = 0$ ✓

(46) wave eqn: $c^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

$u = \cos x \cos ct$
$u_x = -\sin x \cos ct$
$u_{xx} = -\cos x \cos ct$
$u_t = -c \cos x \sin ct$
$u_{tt} = -c^2 \cos x \cos ct$

$c^2 u_{xx} = -\cos x \cos ct\,(c^2)$
$= u_{tt}$ ✓

$u = e^x \cosh ct$
$u_x = e^x \cosh ct$
$u_{xx} = e^x \cosh ct$
$u_t = c e^x \sinh ct$
$u_{tt} = c^2 e^x \cosh t$

$c^2 u_{xy} = c^2 e^x \cosh t$
$= u_{tt}$ ✓

heat eqn: $c \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$

$u = e^{-ct} \sin x$
$u_x = e^{-ct} \cos x$
$u_{xx} = -e^{-ct} \sin x$
$u_t = -c e^{-ct} \sin x$
$c u_{xx} = -c e^{-ct} \sin x$
$= u_t$ ✓

$u = t^{-1/2} e^{-x^2/4ct}$
$u_x = t^{-1/2} \dfrac{-2x}{4ct} e^{-x^2/4ct}$
$u_{xx} = t^{-1/2}\left(\dfrac{-2x}{4ct}\right)\left(\dfrac{-2x}{4ct}\right) e^{-x^2/4ct}$
$+ \; e^{-x^2/4ct}\left(\dfrac{-2t^{-1/2}}{4ct}\right)$
$= \dfrac{4x^2}{16c^2t^2\sqrt{t}} e^{-x^2/4ct} - \dfrac{2t^{-1/2}}{4ct} e^{-x^2/4ct}$
$u_t = t^{-1/2}\left(\dfrac{x^2}{4ct^2}\right) e^{-x^2/4ct} + -\frac{1}{2} t^{-3/2} e^{-x^2/4ct}$
$= \left(\dfrac{x^2}{4ct^2\sqrt{t}} - \dfrac{1}{2t\sqrt{t}}\right) e^{-x^2/4ct}$
$c u_{xx} = \left(\dfrac{x^2}{4ct^2\sqrt{t}} - \dfrac{1}{2t\sqrt{t}}\right) e^{-x^2/4ct}$

$= u_t$ ✓

12.3 HW

(5) $$\lim_{(x,y)\to(-1,2)} \frac{xy - y^3}{(x+y+1)^2} = \lim \frac{p(x,y)}{q(x,y)} = -\frac{5}{2}$$

$$\lim p(x,y) = (-1)(2) - (8) = -10$$
$$\lim q(x,y) = (-1+2+1)^2 = 4$$

(8) $$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2} = \lim \frac{\sin(x^2+y^2)}{(x^2+y^2)} \cdot \frac{1}{\cos(x^2+y^2)}$$

from the example in class, we know

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} = 1$$

So $$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{(x^2+y^2)} = (1)(1) = 1$$

(14) $$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2}$$ use polar coords:

$$\lim_{r\to 0} \frac{r^2\cos\theta\sin\theta\,(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r^2\cos\theta\sin\theta\,\underbrace{(\cos^2\theta - \sin^2\theta)}$$

$$= \lim_{r\to 0} r^2\cos\theta\sin\theta\cos 2\theta = 0 \checkmark$$

35 $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

along x-axis, y = 0 $\Rightarrow \lim_{(x,y)\to(0,0)} \frac{0}{x^2+0} = 0$

$\Rightarrow$ use L'hopital here

along y = x $\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$

the limits are not equal

therefore the limit at the origin DNE

11 $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} r\cos\theta\sin\theta = 0$

12.4 HW 4, 9, 13, 14, 17, 18

(4) $f(x,y) = x^2 y \cos y$

$\nabla f = \langle 2xy\cos y, -x^2 y \sin y + x^2 \cos y \rangle$

(9) $f(x,y,z) = x^2 y e^{x-z}$

$\nabla f = \langle x^2 y e^{x-z} + 2xye^{x-z}, x^2 e^{x-z}, -x^2 y e^{x-z} \rangle$

(13) find ∇f, T $\quad f(x,y) = \cos \pi x \sin \pi y + \sin 2\pi y$

$\nabla f = \langle -\pi \sin \pi x \sin \pi y, \pi \cos \pi x \cos \pi y + 2\pi \cos 2\pi y \rangle$

$\nabla f(-1, \frac{1}{2}) = \langle -\pi \sin(-\pi)\sin(\frac{\pi}{2}), \pi \cos(-\pi)\cos(\frac{\pi}{2}) + 2\pi \cos(\pi) \rangle$

$= \langle 0, -2\pi \rangle$

$z = f(-1, \frac{1}{2}) + \nabla f(-1, \frac{1}{2}) \cdot \langle x+1, y - \frac{1}{2} \rangle$

$= (-1)(1) + 0 + \langle 0, -2\pi \rangle \langle x+1, y-\frac{1}{2} \rangle$

$= -1 - 2\pi y + \pi$

$z = -2\pi y + \pi - 1$

(14) find $\nabla f, T$, $f(x,y) = \frac{x^2}{y}$ $\quad \vec{p} = \langle 2, -1 \rangle$

$\nabla f = \langle \frac{2x}{y}, -\frac{x^2}{y^2} \rangle \quad \nabla f(2,-1) = \langle -4, -4 \rangle$

$f(2,-1) = -4$

$z = T(\vec{p}) = f(2,-1) + \nabla f(2,-1) \cdot \langle x-2, y+1 \rangle$

$z = -4 + \langle -4, -4 \rangle \cdot \langle x-2, y+1 \rangle$

$= -4 - 4x + 8 - 4y - 4 \Rightarrow \boxed{z = -4x - 4y}$

12.4 HW continued

(17) Show that

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

$$\nabla\left(\frac{f}{g}\right) = \frac{\partial\left(\frac{f}{g}\right)}{\partial x}\vec{i} + \frac{\partial\left(\frac{f}{g}\right)}{\partial y}\vec{j} + \frac{\partial\left(\frac{f}{g}\right)}{\partial z}\vec{k}$$

$$= \frac{gf_x - fg_x}{g^2}\vec{i} + \frac{gf_y - fg_y}{g^2}\vec{j} + \frac{gf_z - fg_z}{g^2}\vec{k}$$

$$= \frac{g(f_x\vec{i} + f_y\vec{j} + f_z\vec{k}) - f(g_x\vec{i} + g_y\vec{j} + g_z\vec{k})}{g^2}$$

$$= \frac{g\nabla f - f\nabla g}{g^2}$$ ✓

(18) show $\nabla(f^r) = rf^{r-1}\nabla f$

$$\nabla(f^r) = rf^{r-1}\nabla f$$

$$\nabla(f^r) = \frac{\partial}{\partial x}(f^r)\vec{i} + \frac{\partial}{\partial y}(f^r)\vec{j} + \frac{\partial}{\partial z}(f^r)\vec{k}$$

$$= rf^{r-1}f_x\vec{i} + rf^{r-1}f_y\vec{j} + rf^{r-1}f_z\vec{k}$$

$$= rf^{r-1}\nabla f$$

HW 12.5 4, 10, 18, ~~19~~, 23

④ $f(x,y) = x^2 - 3xy + 2y^2$ $\vec{p} = (-1, 2)$ $\vec{a} = 2\vec{i} - \vec{j}$

find $D_{\vec{u}}f(\vec{p})$

$$\vec{u} = \frac{\langle 2,-1\rangle}{\|\langle 2,-1\rangle\|} = \frac{\langle 2,-1\rangle}{\sqrt{5}} = \frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$$

$$\nabla f = \langle 2x-3y, -3x+4y\rangle$$

$$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u} = \langle 2(-1) - 3(2), -3(-1) + 4(2)\rangle \cdot \vec{u}$$

$$= \langle -8, 11\rangle \cdot \langle \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}\rangle$$

$$= \frac{-16}{\sqrt{5}} - \frac{11}{\sqrt{5}} = \frac{-27}{\sqrt{5}}$$

⑩ $f(x,y) = e^y \sin x$ $\vec{p} = (\frac{5\pi}{6}, 0)$ f $\vec{u}$ (unit vector) where f increases most rapidly + rate

A function increases most rapidly at $\vec{p}$ in direction of the gradient:

$$\nabla f = \langle \cos x\, e^y, \sin x\, e^y\rangle$$

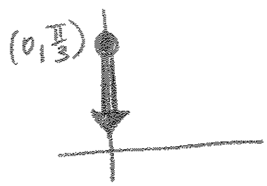
$$\nabla f(\vec{p}) = \langle \cos\tfrac{5\pi}{6}, \sin\tfrac{5\pi}{6}\rangle = \langle -\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\rangle$$

make $\nabla f(\vec{p})$ a unit vector:

$$\vec{u} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle}{\|\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle\|}$$ but $\|\nabla f\| = 1$

so $\vec{u} = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$, rate of change in that direction is 1

(18) Find directional deriv of $f(x,y) = e^{-x}\cos y$ at $(0,\frac{\pi}{3})$ in direction toward origin



the direction will be $\langle 0,-1\rangle$ but need unit vector. $\vec{u} = \langle 0,-1\rangle$ is a unit vector. ✓

$$D_{\vec{u}} f(0,\tfrac{\pi}{3}) = \nabla f(0,\tfrac{\pi}{3})\cdot\vec{u}$$

$$= \left\langle e^{-x}\cos y, -e^{-x}\sin y\right\rangle\Big|_{(0,\frac{\pi}{3})}\cdot\langle 0,-1\rangle$$

$$= \left\langle \cos(\tfrac{\pi}{3}), -\sin(\tfrac{\pi}{3})\right\rangle\langle 0,-1\rangle$$

$$= \sin(\tfrac{\pi}{3})$$

$$= \sqrt{3}/2$$

(19) $T(x,y,z) = \dfrac{200}{5+x^2+y^2+z^2}$

a) where is T the hottest?
T is biggest when $x^2+y^2+z^2$ is smallest ⇒ at origin

b) Find a vector pointing in direction of greatest increase of temp at $(1,-1,1)$

we know this to be $\nabla T(1,-1,1)$

$$\nabla T = \left\langle \frac{-200(2x)}{(5+x^2+y^2+z^2)^2}, \frac{-200(2y)}{(5+x^2+y^2+z^2)^2}, \frac{-200(2z)}{(5+x^2+y^2+z^2)^2}\right\rangle$$

$$\nabla T(1,-1,1) = -\frac{400}{64}\langle 1,-1,1\rangle = -\frac{25}{4}\langle 1,-1,1\rangle$$

vector is $\langle -1,1,-1\rangle$

c) ~~because $(1,-1,1)$ is a point on the sphere,~~
~~$\langle 1,-1,1\rangle$ will point toward the origin~~

$\vec{v} = \langle 0-1, 0+1, 0-1\rangle = \langle -1,1,-1\rangle$ same direction as ∇T.

ex (23) The elevation of a mountain above sea level at the point (x,y) is $f(x,y)$ A mountain climber at $\vec{p}$ notes that the slope in the easterly direction is $-\frac{1}{2}$ and the slope in the notherly direction is $-\frac{1}{4}$. In what direction should he more for faster descent?

want to move in direction of gradient.

$$-\nabla f(\vec{p}) = -\langle f_x(\vec{p}), f_y(\vec{p})\rangle$$

$$\nabla f(\vec{p}) \cdot \langle 1,0\rangle = -\frac{1}{2} \Rightarrow f_x(\vec{p}) = -\frac{1}{2}$$
$$\nabla f(\vec{p}) \cdot \langle 0,1\rangle = -\frac{1}{4} \Rightarrow f_y(\vec{p}) = -\frac{1}{4}$$

$$\Rightarrow -\nabla f(\vec{p}) = -\left\langle -\frac{1}{2}, -\frac{1}{4}\right\rangle = \left\langle \frac{1}{2}, \frac{1}{4}\right\rangle = \frac{1}{4}\langle 2,1\rangle$$

with respect to N and E, how many degrees North from East will he need to go?

$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$ $\quad 90 - \theta = 63.43$

He should move in the direction N 63.43° E