Review for the final

9.6 # 23 find the convergence set for

$$\frac{x-1}{1} + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{4} + \dots$$

the n^{th} term is $\frac{(x-1)^n}{n}$ so check $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \Big/ \frac{(x-1)^n}{n} \right| = \lim_{n\to\infty} |x-1| \left|\frac{n}{n+1}\right|$$

$= |x-1| \cdot 1 \Rightarrow |x-1| < 1$ when $-1 < x-1 < 1 \Rightarrow 0 < x < 2$

Check end points:

at $x=0$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges $\Rightarrow$ 0 included

at $x=2$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ 2 not included

series converges for $0 \leq x < 2$

9.8 # 19 find Taylor series in $x-a$ through $(x-a)^3$ for e^x, $a=1$

$f^{(n)}(x) = e^x$ for all n so that

$f(1) = f'(1) = f''(1) = \dots = e$

$$e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \dots$$

9.9 #14 find the Taylor polynomial of order 3 based at $a=2$ for the function $\sqrt{x}$.

$f(x)=\sqrt{x} \quad f(2)=\sqrt{2}$

$f'(x)=\frac{1}{2}x^{-1/2} \quad f'(2)=\frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}}$

$f''(x)=-\frac{1}{4}x^{-3/2} \quad f''(2)=-\frac{1}{4(\sqrt{2})^3}=-\frac{1}{8\sqrt{2}}=\frac{\sqrt{2}}{16}$

$f'''(x)=\frac{3}{8}x^{-5/2} \quad f'''(3)=\frac{3}{8(\sqrt{2})^5}=\frac{3}{32\sqrt{2}}=\frac{3\sqrt{2}}{64}$

$P_3(x)=\sqrt{2}+\frac{\sqrt{2}}{4}(x-2)-\frac{\sqrt{2}}{32}(x-2)^2+\frac{\sqrt{2}}{128}(x-2)^3$

10.3 #24 Sketch the graph of the given eqn:

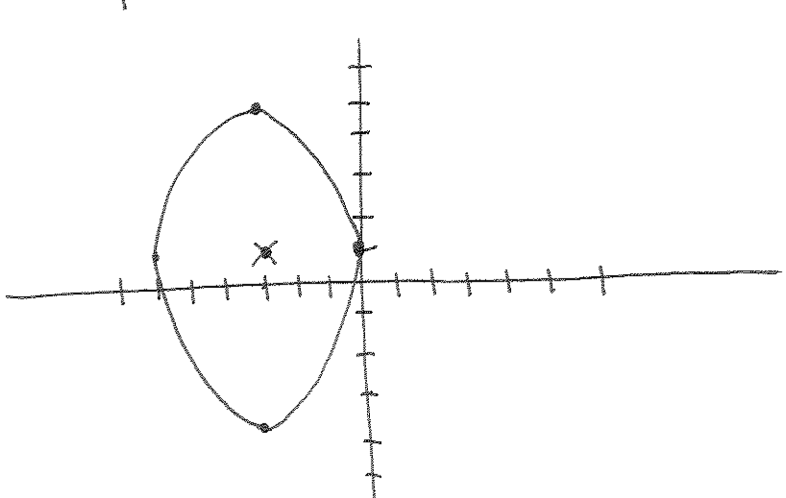
$25x^2+9y^2+150x-18y+9=0$

$25(x^2+6x+\quad)+9(y^2-2y+1)=0$

$25(x^2+6x+9)+9(y-1)^2=225$

$25(x+3)^2+9(y-1)^2=225$

$\frac{(x+3)^2}{9}+\frac{(y-1)^2}{25}=1$ ELLIPSE centered at $(-3,+1)$



10:4 # 28 find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ without eliminating the parameter

$x = \cot t - 2$, $y = -2\csc t + 5$ $\quad 0 < t < \pi$

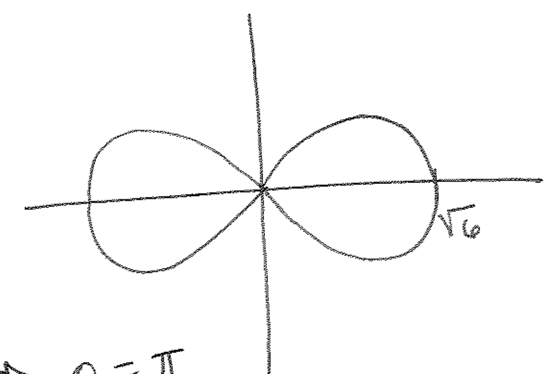
$$\left.\begin{aligned} \frac{dx}{dt} &= -\csc^2 t \\ \frac{dy}{dt} &= 2\csc t \cot t \end{aligned}\right\} \frac{dy}{dx} = \frac{2\csc t \cot t}{-\csc^2 t} = \frac{2\cot t}{-\csc t} = -2\cos t$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = 2\sin t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{2\sin t}{-\csc^2 t} = -2\sin^3 t$$

10.7 # 8 sketch the graph & find the area of the region bounded by it: $r^2 = 6\cos 2\theta$

$\theta = 0 \Rightarrow r^2 = 6 \Rightarrow r = \sqrt{6}$



find the tangent line at the pole by setting $r = 0$

$0 = 6\cos 2\theta \Rightarrow \cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$

$$A = 4 \cdot \frac{1}{2}\int_0^{\pi/4} 6\cos 2\theta \, d\theta$$

$$= 12\left[\frac{\sin 2\theta}{2}\right]\Big|_0^{\pi/4} = 6\sin\frac{\pi}{2} = 6$$

$A = 6$

11.3 #31 find $\text{proj}_{\vec{u}}\vec{w}$ if $\vec{u} = \langle 3,2,1\rangle$ $\vec{w} = \langle 1,5,-3\rangle$

$$\text{proj}_{\vec{u}}\vec{w} = \frac{\vec{w}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$$

$\langle 3,2,1\rangle \cdot \langle 1,5,-3\rangle = 3+10-3 = 10$

$\|\vec{u}\|^2 = 3^2+2^2+1^2 = 14$

$$= \frac{10}{14}\vec{u}$$

$$= \frac{10}{14}\langle 3,2,1\rangle = \frac{5}{7}\langle 3,2,1\rangle$$

11.3 #43. Which of the following do not make sense?

a) $\vec{u}\cdot\underbrace{(\vec{v}\cdot\vec{w})}_{\text{scalar}}$ does not make sense

b) $\underbrace{(\vec{u}\cdot\vec{w})}_{\text{scalar}}+\underbrace{\vec{w}}_{\text{vector}}$ does not make sense

c) $\|\vec{u}\|(\vec{v}\cdot\vec{w})$ OK scalar * scalar

d) $(\vec{u}\cdot\vec{v})\vec{w}$ ok scalar * vector

11.4 #8 find the area of the parallelogram with $\vec{a} = 2\vec{i}+2\vec{j}-\vec{k}$ and $\vec{b} = -\vec{i}+\vec{j}-4\vec{k}$ as adjacent sides

area $= \|\vec{a}\times\vec{b}\| \Rightarrow$ $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix}$

$$= \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}\vec{k} = -7\vec{i}+9\vec{j}+4\vec{k}$$

$$\|\vec{a}\times\vec{b}\| = \sqrt{49+81+16} = 146$$

11.5 #26 find velocity, acceleration + speed at the indicated time:

$\vec{r}(t) = \sin 2t\vec{i} + \cos 3t\vec{j} + \cos 4t\vec{k} \quad t_1 = \frac{\pi}{2}$

$\vec{v}(t) = 2\cos 2t\,\vec{i} - 3\sin 3t\,\vec{j} - 4\sin 4t\,\vec{k}$

$\vec{a}(t) = -4\sin 2t\,\vec{i} - 9\cos 3t\,\vec{j} - 16\cos 4t\,\vec{k}$

$\vec{v}(\frac{\pi}{2}) = -2\vec{i} + 3\vec{j} \quad \vec{a}(\frac{\pi}{2}) = -16\vec{k}$

$s(\frac{\pi}{2}) = \|\vec{v}(\frac{\pi}{2})\| = \sqrt{4+9} = \sqrt{13}$

11.6 #16 find symmetric eqns of the line through $(2,-4,5)$ parallel to the plane $3x+y-2z=5$ and perpendicular to the line $\frac{x+8}{2} = \frac{y-5}{3} = \frac{z-1}{-1}$

$\langle 2,3,-1\rangle$ is the direction vector $\perp$ to the line we seek
want a vector $\perp$ to $\langle 2,3,-1\rangle$ and $\perp$ to the normal of the plane $3x+y-2z=5$ which is $\langle 3,1,-2\rangle$

$$\vec{v} = \langle 3,1,-2\rangle \times \langle 2,3,-1\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 2 & 3 & -1 \end{vmatrix} = \langle 5,-1,7\rangle$$

$\vec{v}$ and the point $(2,-4,5)$ give

$x = 2+5t, \quad y = -4-t, \quad z = 5+7t$

$$\boxed{\frac{x-2}{5} = \frac{y+4}{-1} = \frac{z-5}{7}}$$

11.7. ex 7 find $\vec{T}, \vec{N}, \vec{B}$ for uniform circular motion

$\vec{r}(t) = a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}$

see page 600

12.4 # 11 find the gradient and the equation of the tangent plane at $\vec{p} = (-2,3)$ if $f(x) = x^2y - xy^2$

$\nabla f = \langle 2xy - y^2, x^2 - 2xy \rangle$ $\quad \nabla f(-2,3) = \langle -21, 16 \rangle$

$z = f(-2,3) + \nabla f(-2,3) \cdot \langle x+2, y-3 \rangle$

$= 30 + (-21x - 42 + 16y - 48)$

$= -21x + 16y - 60$

12.5 #17 find the directional derivative of $f(x,y,z) = xy + z^2$ at $(1,1,1)$ in the direction towards $(5,-3,3)$

$(5,-3,3) - (1,1,1)$ gives the vector $\vec{a} = \langle 4, -4, 2 \rangle$

$\|\vec{a}\| = \sqrt{16+16+4} = 6$

$\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$= \langle y, x, 2z \rangle \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

$= \frac{2}{3}y - \frac{2}{3}x + \frac{2}{3}z$

$D_{\vec{u}} f(1,1,1) = \frac{2}{3}$

12.6 #7 find $\frac{\partial w}{\partial t}$ by using the chain rule if

$w = x^2 y$, $x = st$, $y = s - t$

express answer in terms of s & t.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2xy)s + x^2(-1) = 2\left(st(s-t)(s)\right) + s^2t^2(-1)$$

$$= (2s^2t - 2st^2)s + s^2t^2(-1)$$

$$= 2s^3t - 2s^2t^2 - s^2t^2$$

$$= 2s^3t - 3s^2t^2$$

(16) Show $z = x^2 y$ and $y = \frac{1}{4}x^2 + \frac{3}{4}$ intersect at $(1,1,1)$ and have perp tangent planes there.

if they intersect then

$$z = x^2 y = 1$$

$$\left.\begin{array}{l} x^2 y = 1 \\ \frac{1}{4}x^2 + \frac{3}{4} - y = 0 \end{array}\right\} \quad x^2 y - 1 \stackrel{?}{=} \frac{1}{4}x^2 + \frac{3}{4} - y$$

$$(1)^2(1) - 1 = \frac{1}{4}(1) + \frac{3}{4} - 1$$

$$0 = 0 \checkmark$$ they intersect.

tangent planes:

$$z = x^2 y \left\{ \begin{array}{l} z - z_0 = 2xy(x - x_0) + x^2(y - y_0) \\ z - 1 = 2(x-1) + y - 1 \\ \boxed{2x + y - z = 0} \end{array} \right.$$ normal$_1$ $\langle 2, 1, -1 \rangle$

$y = \frac{1}{4}x^2 + \frac{3}{4}$ is a parabola in the x-y plane and in 3D is

if $z = f(x,y) \Rightarrow F(x,y,z) = f(x,y) - z = 0$

then $y = f(x) \Rightarrow F(x,y) = f(x) - y = 0$

$$F(x,y) = \frac{1}{4}x^2 + \frac{3}{4} - y$$

$$0 - 1 = \frac{1}{2}(x-1) - 1(y-1)$$

$$\boxed{\frac{1}{2}x - y = -1}$$ normal$_2$ $\langle \frac{1}{2}, -1, 0 \rangle$

$\vec{n}_1 \cdot \vec{n}_2 = \langle 2, 1, -1 \rangle \cdot \langle \frac{1}{2}, -1, 0 \rangle = 1 - 1 + 0 = 0 \checkmark$ perpendicular

13.3 #7 evaluate the iterated integral

$$\int_{1/2}^{1}\int_{0}^{2x} \cos \pi x^2 \, dy \, dx$$

Let $u = \pi x^2$

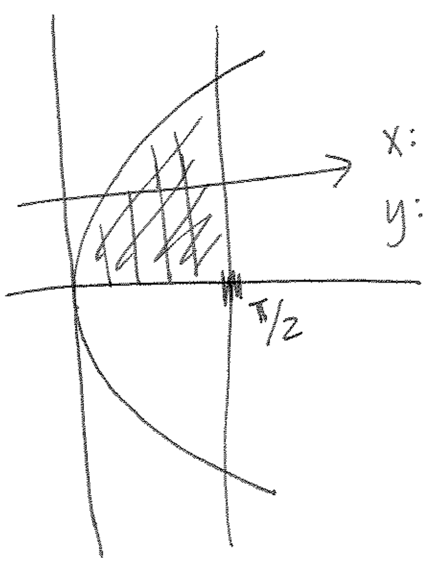
$du = 2\pi x \, dx$

$$= \int_{1/2}^{1} \left[\cos(\pi x^2)\, y\right]\Big|_0^{2x} dx = \int_{\frac{1}{2}}^{1} 2x \cos \pi x^2 dx$$

$$= \frac{1}{\pi}\int_{\frac{\pi}{4}}^{\pi} \cos u \, du = \frac{1}{\pi}\left[\sin u\right]_{\frac{\pi}{4}}^{\pi} = \frac{1}{\pi}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2\pi}$$

13.3 reverse the order of integration

$$\int_0^{\sqrt{\pi/2}}\int_{y^2}^{\pi/2} f(x,y)\,dx\,dy$$



$x: y^2 \to \pi/2$

$y: 0 \to \sqrt{\pi/2}$

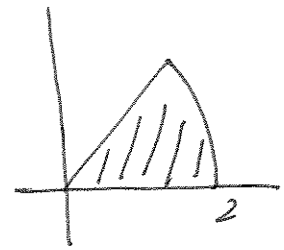
$y: 0 \to \sqrt{x}$

$x: 0 \to \pi/2$

$$\int_0^{\pi/2}\int_0^{\sqrt{x}} f(x,y)\,dy\,dx$$

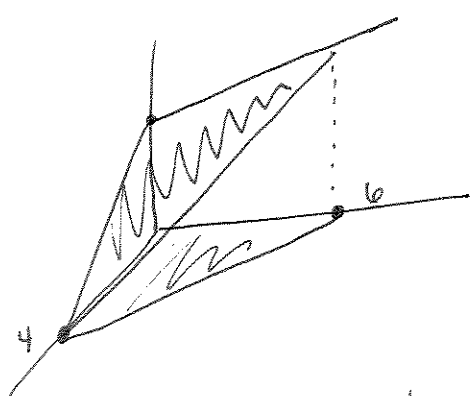
13.4 # 13 sketch the region and the area:

$$\int_0^{\pi/4}\int_0^{2} r\,dr\,d\theta$$



$$= \int_0^{\pi/4} \frac{r^2}{2}\Big|_0^2 d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

13.6 # 2 find the area of the part of the plane $3x-2y+6z=12$ that is bounded by the planes $x=0, y=0$, and $3x+2y=12$



$y = 6 - \frac{3}{2}x$ in x-y plane

$z = \frac{12+2y}{6} = 2 + \frac{1}{3}y$ in z-y plane

$6z$

$z = 2 - \frac{1}{2}x + \frac{1}{3}y$ $\quad f_x = -\frac{1}{2}\quad f_y = \frac{1}{3}$

$$SA = \int_0^4\int_0^{-3/2x+6} \sqrt{\frac{1}{4}+\frac{1}{9}+1}\,dy\,dx = \frac{7}{6}\int_0^4 6-\frac{3}{2}x\,dx$$

$$= \frac{7}{6}\left[6x - \frac{3x^2}{4}\right]_0^4 = \frac{7}{6}\left[24-12\right] = 7\cdot 2 = 14$$

13:9 # 16 find the transformation from the u-v plane to the x-y plane + find the Jacobian

$u = x^2 \quad v = xy$

$x = \sqrt{u}$

$y = (\sqrt{u})^{-1} v = \frac{v}{\sqrt{u}}$

$$J(u,v) = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ \frac{-v}{2u^{3/2}} & \frac{1}{\sqrt{u}} \end{vmatrix} = \frac{1}{2u}$$

14.1 # 13 find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ if $\vec{F} = x^2 \vec{i} - 2xy \vec{j} + yz^2 \vec{k}$

$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle x^2, -2xy, yz^2 \rangle$

$= 2x - 2x + 2yz = 2yz$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xy & yz^2 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2xy & yz^2 \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 & yz^2 \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & -2xy \end{vmatrix} \vec{k}$$

$= z^2 \vec{i} + 0\vec{j} - 2y\vec{k}$

$= \langle z^2, -2y \rangle$

14.2 # 7 evaluate $\int_C y\,dx + x^2 dx$ c is the curve $x = 2t$, $y = t^2 - 1$, $0 \le t \le 2$

$dx = 2dt$

$dy = 2t\,dt$

$\Rightarrow \int_0^2 2(t^2-1) + (2t^2)(2t)dt = \int_0^2 4t^2 - 2 + 8t^3 dt = \left[\frac{4t^3}{3} - 2t + \frac{8t^4}{4} \right]_0^2$

$= \frac{100}{3}$

14.3 #8 determine if $\vec{F}$ is conservative. If so, find $f \ni \nabla f = \vec{F}$

$$\vec{F}(x,y) = -e^{-x}\ln y\,\vec{i} + \frac{e^{-x}}{y}\vec{j}$$

$$\frac{\partial M}{\partial y} = \frac{-e^{-x}}{y} = \frac{\partial N}{\partial x} \quad \text{Conservative}$$

$$\frac{\partial f}{\partial x} = e^{-x}\ln y \Rightarrow f(x,y) = -e^{-x}\ln(y) + C(y)$$

$$\frac{\partial f}{\partial y} = \frac{e^{-x}}{y} + C'(y) \Rightarrow C'(y) = 0 \Rightarrow C(y) = c$$

$$f(x,y) = -e^{-x}\ln y + c$$

14.3 Find $\int_C \vec{F}\cdot d\vec{r}$ where C is any smooth curve from $(0,1)$ to $(1,2)$

and $\vec{F}(x,y) = (2x+5)\vec{i} + (2y-4)\vec{j}$

$$\frac{\partial f}{\partial x} = 2x+5 \Rightarrow f(x,y) = x^2 + 5x + c(y)$$

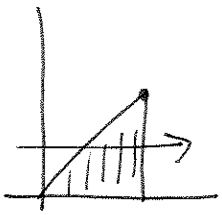
$$\frac{\partial f}{\partial y} = C'(y) = 2y-4 \Rightarrow c(y) = y^2 - 4y + c$$

$$\Rightarrow f(x,y) = x^2 + 5x + y^2 - 4y + c$$

so that $\int_C \nabla f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$

$$\left[x^2 + 5x + y^2 - 4y + c\right]_{(0,1)}^{(1,2)} = \left[1+5+4-8+c-1+4-c\right] = 5$$

14.4 Use Green's theorem to evaluate $\oint 2xy\,dx + (e^x + x^2)dy$
where C is the bdry of the triangle w/ vertices (0,0) (1,0) (1,1)



$M = 2xy$
$N = e^x + x^2$

$$\oint 2xy\,dx + e^x + x^2\,dy$$

$$= \iint_S e^x + 2x - 2x\,dA$$

x: $y \to 1$
y: $0 \to 1$

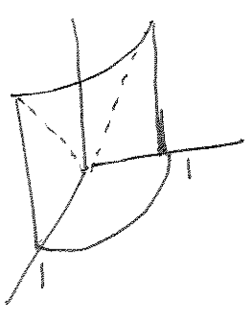
$$= \iint_S e^x\,dA = \int_0^1 \int_y^1 e^x\,dx\,dy$$

$$= \int_0^1 e^1 - e^y\,dy = \left[ey - e^y\right]_0^1 = e - e - 0 + 1$$

$$= 1$$

14.5 #12 calculate the flux of $\vec{F}$ across G where
$\vec{F}(x,y,z) = 2\vec{i} + 5\vec{j} + 3\vec{k}$ G is the part of the cone $z = (x^2+y^2)^{1/2}$
inside the cylinder x^2+y^2+1

want to use $\iint_R -Mf_x - Nf_y + P\,dA$



$$= \iint_R -2x(x^2+y^2)^{-1/2} - 5y(x^2+y^2)^{-1/2} + 3\,dA$$

in polar: $\int_0^{2\pi}\int_0^1 \left(\frac{-2r\cos\theta - 5r\sin\theta}{r} + 3\right) r\,dr\,d\theta$

$$= \int_0^{2\pi} (-2\cos\theta - 5\sin\theta + 3)\,d\theta \int_0^1 r\,dr$$

$$= \left[-2\sin\theta + 5\cos\theta + 3\theta\right]_0^{2\pi}\left(\frac{1}{2}\right) = \frac{6\pi}{2} = 3\pi$$