MATHEMATICS 3210-2. Homework 5: Solutions.

September 11, 2000

1. Problem # 1, page 37 from the textbook. Prove each of the following limits:

a) $\lim_{n\to\infty}(3 + 1/n) = 3$.

Solution. In order to prove this limit, given $\epsilon > 0$ we need to find $n_0 \in \mathbb{N}$ so that for $n \geq n_0$, $|3 + 1/n - 3| < \epsilon$. Note that, $|3 + 1/n - 3| = 1/n$. $1/n < \epsilon \iff n > 1/\epsilon$. By the Archimedian Principle there exists $n_0 \in \mathbb{N}$ so that $n_0 > 1/\epsilon$. Hence for all $n \geq n_0$ we have $n \geq n_0 > 1/\epsilon$, and $1/n < \epsilon$. □

b) $\lim_{n\to\infty} 2(1 - 1/n) = 2$.

Solution. $|2(1 - 1/n) - 2| = 2/n$. By the Archimedian Principle there exists $n_0 \in \mathbb{N}$ so that $n_0 > 2/\epsilon$. Hence for all $n \geq n_0$ we have $n \geq n_0 > 2/\epsilon$, and $2/n < \epsilon$. □

c) $\lim_{n\to\infty}(5 + n)/n^2 = 0$.

Solution. $|(5+n)/n^2 - 0| = \frac{5}{n^2} + \frac{1}{n}$. Given $\epsilon > 0$ we need to find n_1 so that for $n \geq n_1$, $\frac{5}{n^2} + \frac{1}{n} \leq \epsilon$. As in (a), by the Archimedian Principle there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ we have:

$$1/n < \epsilon/2. \tag{1}$$

Note that for $n > 5$, $5/n < 1$ and thus $\frac{5}{n^2} < 1/n$. Therefore for $n \geq \max(5, n_0)$,

$$\frac{5}{n^2} < \epsilon/2. \tag{2}$$

By adding the inequalities (1) and (2) we get

$$\frac{5}{n^2} + \frac{1}{n} < \epsilon$$

for all $n \geq n_1 = \max(5, n_0)$. □

d) $\lim_{n\to\infty} \pi - 3/\sqrt{n} = \pi$.

Solution. $|\pi - (\pi - 3/\sqrt{n})| = 3/\sqrt{n}$. Given $\epsilon > 0$ we need to find n_0 so that for $n \geq n_0$, $3/\sqrt{n} < \epsilon$. Note that the last inequality is equivalent to $9/n < \epsilon^2$, which in turn is equivalent to $n > 9/\epsilon^2$. By the

Archimedian Principle there exists $n_0 \in \mathbb{N}$ so that $n_0 > 9/\epsilon^2$. Hence for all $n \geq n_0$ we have $n > 9/\epsilon^2$, equivalently, $3/\sqrt{n} < \epsilon$. $\square$

2. Problem # 3, page 37, from the textbook.

a) Prove that the sequence $\{(-1)^n\}$ contains subsequences which converge and subsequences which do not converge.

Solution. By choosing $n = 2k$ (the even natural numbers) we get the subsequence $\{(-1)^{2k}\} = \{1\}$. This is a constant sequence, therefore it converges.

By Example 2.3, the sequence $\{(-1)^n\}$ does not have a limit. This sequence is a subsequence of itself, therefore we get a subsequence which does not converge.

b) Find a convergent subsequence of $x_n = n + (-1)^{3n}n$.

Solution. Let $n = 2k+1$ be an odd number, then $3n$ is again an odd number. (Indeed, $3n = 2n+n$. The number $2n$ is even and the number n is odd. The sum of an even and odd number is odd, thus $2n + n$ is odd.) Thus for odd n we have $(-1)^{3n} = -1$. Hence the subsequence corresponding to odd indices equals $x_{2k+1} = n - n = 0$. This is a constant sequence, therefore it converges (to zero in this case). $\square$