MATHEMATICS 3210-2. Homework 4: Solutions.

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1. Problem # 1, page 23, from the textbook. Find infimum and supremum of each of the following sets:

a) $E = \{4, 3, 2, 1, 8, 7, 6, 5\}$.

Solution. By inspection, each number from E is bounded from above by 8. To check that $8 = sup(E)$ we use the approximation property:

For each $\epsilon > 0$, $(8 - \epsilon, 8] \cap E \neq \emptyset$, since this intersection contains 8.

Similarly, each number in E is bounded from below by 1 and by the same as above approximation principle, $1 \in [1, 1 + \epsilon) \cap E \neq \emptyset$, implies that $1 = \inf(E)$. $\square$

b) $E = \{x \in \mathbb{R} | x^2 - 3x - 5 = 0\}$.

Solution. Solving the quadratic equation $x^2 - 3x - 5 = 0$ we find two solutions:

$$x_1 = (3 + \sqrt{9 + 20})/2 = (3 + \sqrt{29})/2$$

$$x_2 = (3 - \sqrt{9 + 20})/2 = (3 - \sqrt{29})/2.$$

Since $\sqrt{29} > 0$ and $-\sqrt{29} < 0$, we have $x_2 < x_1$. This implies that x_1 is an upper bound for E and applying the approximation property as in (a) we get: $x_1 = sup(E)$.

To check that $x_2 = \inf(E)$ note that x_2 is a lower bound for E and since $x_2 \in E$, $x_2 = \inf(E)$. $\square$

c) $E = [a, b)$ where $a < b$ are real numbers.

Solution. Since $E = \{x : a \leq x < b\}$, the number b is an upper bound for E. To see that $b = sup(E)$ we use the approximation property:

Let $y < b$. If $y \leq a$ then $(y, b] \cap [a, b) = (a, b)$ contains $m = (a + b)/2$ and is therefore nonempty:

$$y \leq a = (a + a)/2 < (a + b)/2 = m, \quad m = (a + b)/2 < (b + b)/2 = b.$$

If $y > a$ then $(y, b] \cap [a, b) = (y, b)$ contains $m = (y + b)/2$ and is therefore nonempty as well:

$$a < y = (y + y)/2 < (y + b)/2 = m, \quad m = (y + b)/2 < (b + b)/2 = b.$$

In both cases the approximation principle implies that $b = sup(E)$.

We now check that $a = \inf(E)$. This is actually easier, since a is a lower bound for E and since $a \in E$, the approximation principle implies that $a = \inf(E)$.

(d) $E = \{p/q \in \mathbb{Q} | p^2 < 2q^2 \text{ and } p, q > 0\}$.

Solution. We first find the infimum of E. Since $p, q > 0$, $p/q > 0$ and thus 0 is a lower bound for E. By the density of rational numbers, each open interval $(0, \epsilon)$ contains a rational number p/q. Any such number will belong to E provided that $\epsilon < \sqrt{2}$. If $\epsilon > \sqrt{2} > 1$ then $(0, \epsilon)$ contains rational number $1 \in E$. Thus by the approximation principle, $0 = \inf(E)$.

I claim that $\sqrt{2} = \sup(E)$. Indeed, if $p^2 < 2q^2$ then $(p/q)^2 < 2$ and thus $p/q < \sqrt{2}$. This implies that $\sqrt{2}$ is an upper bound for E. Let's check that this is the least upper bound. Pick $\epsilon > 0$. If $\epsilon > \sqrt{2}$, then $(\sqrt{2} - \epsilon, \sqrt{2}]$ contains the rational number 1. If $\epsilon < \sqrt{2}$, then $(\sqrt{2} - \epsilon, \sqrt{2}) \cap \mathbb{Q} \subset E$ and by density property of rational numbers, $(\sqrt{2} - \epsilon, \sqrt{2})$ contains a rational number which is therefore an element of E. Thus the approximation property implies that $\sqrt{2} = \sup(E)$. ☐

(e) $E = \{x \in \mathbb{R} | x = 1 + (-1)^n, \text{ for } n \in \mathbb{N}\}$.

Solution. If $n \in \mathbb{N}$ is even then $x = 1 + (-1)^n$ equals $1 + 1 = 2$. If n is odd then $x = 1 + (-1)^n$ equals $1 - 1 = 0$. Thus $E = \{0, 2\}$. Then 0 is a lower bound for E and 2 is an upper bound for E. Since $0 \in E$ and $2 \in E$, the approximation principle implies that $0 = \inf(E)$, $2 = \sup(E)$. ☐

(f) $E = \{x \in \mathbb{R} | x = 1/n - (-1)^n, \text{ for } n \in \mathbb{N} - \{0\}\}$.

Solution. If n is even then $x = 1/n - (-1)^n = 1/n - 1$. If n is odd then $x = 1/n - (-1)^n = 1/n + 1$. Note that

$$1/n - 1 \leq 1 + 1/n, \text{ for all } n \in \mathbb{N} - \{0\}.$$

On the other hand, $1 + 1/n \leq 1 + 1/1 = 2$, for all $n \in \mathbb{N} - \{0\}$. Thus 2 is an upper bound for E and since $2 \in E$ (note that 1 is an odd number), the approximation principle implies that $2 = \sup(E)$. We now find $\inf(E)$. Note that $1 + 1/n \geq 1/n - 1 > -1$ for all n. Thus -1 is a lower bound for E. To check that $-1 = \inf(E)$ we again use the approximation property. Let $\epsilon > 0$. By the Archimedian principle,

$$2k > 1/\epsilon$$

for some $k \in \mathbb{N}$. Thus $\epsilon > \frac{1}{2k}$ and

$$-1 < -1 + \frac{1}{2k} < -1 + \epsilon$$

and $-1 + \frac{1}{2k} \in [-1, -1 + \epsilon)$. Since $-1 + \frac{1}{2k} \in E$, the intersection $E \cap [-1, -1 + \epsilon)$ is nonempty. Therefore by the approximation property, $-1 = \inf(E)$. ☐

(g) $E = \{1 + (-1)^n/n : n \in \mathbb{N} - \{0\}\}$.

Solution. If n is even then $x = 1 + (-1)^n/n = 1 + 1/n$. If n is odd then $x = 1 + (-1)^n/n = 1 - 1/n$. Note that

$$1 - 1/n < 1 < 1 + 1/n$$

for all n. If $n \in \mathbb{N} - \{0\}$ is even then $n \geq 2$, thus $1 + 1/n \leq 1 + 1/2 = 1.5$. Thus 1.5 is an upper bound for E. It follows from the approximation property that $1.5 = \sup(E)$ (since $1.5 \in E$).

If $n \in \mathbb{N}$ is odd then $n \geq 1$, thus $1 - 1/n \geq 1 - 1 = 0$. Hence $0 \in E$ is a lower bound for E and the approximation principle implies that $0 = \inf(E)$. □

2. Problem # 9, page 24, from the textbook. Prove that for each $a, b \in \mathbb{R}$ such that $0 \leq a < b$, there exists $n, m \in \mathbb{N}$ such that $a < m/10^n < b$.

Solution. We first show that for each $x > 0$ there exists n so that $10^n > x$. Suppose not. Then the nonempty set $E = \{10^n : n \in \mathbb{N}\}$ is bounded from above by x. Hence there exists a supremum, $y = \sup(E)$. Since $y \geq 10^n$ for all n, $y \geq 10$. By the approximation property, $(y-1, y]$ contains 10^n for some n. Thus $y - 1 < 10^n$,

$$10^{n+1} = 10 \cdot 10^n > 10 \cdot (y - 1) = y + (9y - 10) \geq y + 90 - 10 > y.$$

Thus y is not an upper bound for E and we reached a contradiction.

Therefore, for each $x > 0$ there exists n so that $10^n > x$. Hence for each $\epsilon = 2/x > 0$ there exists n so that $10^n > 2/\epsilon$. Equivalently, $\epsilon \cdot 10^n > 2$.

The length of the interval (a, b) equals $\epsilon = b - a$. Thus we can find $n \in \mathbb{N}$ such that $\epsilon \cdot 10^n > 2$. Therefore the length of the interval $(10^n a, 10^n b)$ equals $10^n \epsilon$ and is greater than 2. Thus, by the "density property" of the integers proven in the class, the interval $(10^n a, 10^n b)$ contains an integer m:

$$10^n a < m < 10^n b.$$

Since $10^n a \geq 0$, $m \in \mathbb{N}$. Thus the interval (a, b) contains $m/10^n$:

$$a < m/10^n < b. \quad \square$$