

MATHEMATICS 3210-2. Homework 3

Solutions.

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1. Problem # 6 (c), page 12, from the textbook:

Prove that sum of rational and irrational number is always irrational.

What about the product rational·irrational ? When is it rational and when it is irrational?

Solution. Let x be an irrational number and p/q be a rational number (where $p, q \in \mathbb{Z}$). Suppose that $x + p/q$ is a rational number equal to s/r where $s, r \in \mathbb{Z}$. Then

$$x = s/r - p/q = \frac{sq - pr}{rq}.$$

Since product and difference of integers is again an integer, it follows that $sq - pr \in \mathbb{Z}$ and $rq \in \mathbb{Z}$. Thus x is a rational number. Contradiction.

Now suppose that $x \cdot \frac{p}{q}$ is a rational number s/r . If $p/q \neq 0$ we get:

$$x = \frac{s \cdot q}{r \cdot p}$$

is a fraction of two integers and therefore x is a rational number, contradicting our assumptions. On the other hand, if $p/q = 0$ then $x \cdot \frac{p}{q} = 0$, which is a rational number. Thus $x \cdot \frac{p}{q}$ is irrational provided that $p/q \neq 0$. If p/q is zero then the product $x \cdot \frac{p}{q}$ is rational. \square

2. Problem # 6, parts 1 and 2, page 17, from the textbook:

Using induction prove that:

- (a) $n < 2^n$,
- (b) $n^2 \leq 2^n + 1$,

for all positive integers $n = 1, 2, 3, \dots$

Solution. (a) We prove the inequality by induction.

(1) For $n = 1$ we get $1 < 2^1 = 2$ which is a valid inequality.

(2) Suppose that the inequality holds for k , i.e.

$$k < 2^k.$$

We will prove that it holds for $k + 1$ as well, that is:

$$k + 1 < 2^{k+1}.$$

Indeed,

$$1 = 1^k = \underbrace{1 \cdot \dots \cdot 1}_{k \text{ times}} < \underbrace{2 \cdot \dots \cdot 2}_{k \text{ times}} = 2^k$$

for $k \geq 1$, since $1 < 2$. Now, adding the inequalities:

$$1 < 2^k, \quad k < 2^k$$

we get

$$1 + k < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Thus, by induction, the inequality (a) is proven.

(b) We first verify this inequality for $n = 1$, $n = 2$ and $n = 3$:

$$1^2 = 1 \leq 2^1 + 1 = 3$$

is correct. For $n = 2$,

$$2^2 = 4 \leq 2^2 + 1 = 5$$

is correct as well. Lastly, for $n = 3$

$$3^2 = 9 \leq 2^3 + 1 = 8 + 1 = 9$$

is also valid. For $n \geq 4$ we verify the inequality (b) by induction. Suppose that

$$k^2 \leq 2^k + 1, \quad \text{for some } k \geq 3.$$

We will prove that

$$(k + 1)^2 \leq 2^{k+1} + 1, \quad (\text{here of course } k \geq 3).$$

This inequality is equivalent to:

$$k^2 + 2k + 1 \leq 2^k + 2^k + 1.$$

Since by the induction hypothesis $k^2 \leq 2^k + 1$, it suffices to show that

$$2k + 1 \leq 2^k.$$

Recall that by the induction hypothesis

$$k^2 - 1 \leq 2^k.$$

If $2k + 1 \leq k^2 - 1$, then $2k + 1 \leq k^2 - 1 \leq 2^k$ and we will be done. Thus we will prove that $2k + 1 \leq k^2 - 1$ for $k \geq 3$. This inequality is equivalent to

$$k^2 - 2k - 2 \geq 0.$$

Note that the left hand side is “almost” a complete square, so we will rewrite this inequality as

$$(k - 1)^2 - 3 \geq 0.$$

The latter is equivalent to $(k - 1)^2 \geq 3$. However, since $k \geq 3$, $k - 1 \geq 2$ and $(k - 1)^2 \geq 4$. Since $4 > 3$, the inequality $(k - 1)^2 \geq 3$ holds. Thus the equivalent inequality $2k + 1 \leq k^2 - 1$ holds as well and we are done. \square

An alternative solution. Note that in the proof we needed the inequality

$$2n + 1 \leq 2^n, \quad n \geq 3.$$

Let us prove this inequality by induction. For $n = 3$ we get:

$$7 = 6 + 1 \leq 8$$

which is a valid inequality. Suppose that $2k + 1 \leq 2^k$. We will prove that

$$2(k + 1) + 1 \leq 2^{k+1}.$$

This inequality is equivalent to

$$2k + 1 + 2 \leq 2^k + 2^k$$

Note that $2k + 1 \leq 2^k$ by the induction hypothesis. On the other hand,

$$2 \leq \underbrace{2 \cdot \dots \cdot 2}_{k \text{ times}} = 2^k,$$

for all $k \geq 1$. By adding the inequalities $2k + 1 \leq 2^k$ and $2 \leq 2^k$ we get the required inequality

$$2k + 1 + 2 \leq 2^k + 2^k. \quad \square$$