MATHEMATICS 3210-2. Homework 2. Solutions.

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1. Problem # 1 from the handout. For $f : A \to B$, compute $f(A_0), f^{-1}(B_0)$.

(a) $f(x) = x^2$, $f : A = \{-1, 0, 1\} \to B = \mathbb{R}$. $A_0 = \{-1, 1\}$, $B_0 = \{0, 1\}$.

Solution. $f(A_0) = \{f(-1), f(1)\}$. Since $f(-1) = (-1)^2 = 1, f(1) = 1^1 = 1$ we get:

$$f(A_0) = \{f(-1), f(1)\} = \{1, 1\} = \{1\}.$$

To compute $f^{-1}(B_0)$ note that this set equals $\{x \in A : f(x) \in B_0\} = \{x \in A : f(x) = 0\} \cup \{x \in A : f(x) = 1\}$. We have:

$$\{x \in A : f(x) = 0\} = \{x \in \{-1, 0, 1\} | x^2 = 0\} = \{0\}.$$

$$\{x \in B : f(x) = 1\} = \{x \in \{-1, 0, 1\} | x^2 = 1\} = \{1, -1\}.$$

Hence $f^{-1}(B_0) = \{0\} \cup \{1, -1\} = \{-1, 0, 1\}$. □

(b) $A = B = \mathbb{R}$, $A_0 = \{x \in \mathbb{R} | x > 0\}$, $B_0 = \{0\}$,

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Solution. To compute $f(A_0)$ note that $A_0 \subset \{x | x \geq 0\}$, hence the restriction of f to A_0 is given by $f(x) = x^2$. Square of each $x > 0$ is a positive real number. On the other hand, each positive real number y is the square of another positive real number, namely, $\sqrt{y}$. Hence $f(A_0) = \{x \in \mathbb{R} | x > 0\}$.

To find $f^{-1}(B_0)$ note that this set equals $\{x \in \mathbb{R} : x^2 = 0\} = \{0\}$. □

(c) $A = B = \mathbb{R}$, $A_0 = B_0 = (-2, 1) = \{x \in \mathbb{R} | -2 < x < 1\}$,

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x < 0 \end{cases}$$

Solution. The function f is given by three different formulae on three different subsets of the domain. To compute $f(A_0)$ we break the set A_0 into the subsets: $A_0 = (-2,0) \cup \{0\} \cup (0,1)$ according to the definition of the function f so that on each of these subsets the function f is given by single formula. Then $f(A) = f((-2,0)) \cup f(\{0\}) \cup f((0,1))$.

$$f((-2,0)) = \{-1\}, f(\{0\}) = \{0\}, f((0,1)) = \{1\}.$$

Hence $f(A) = \{-1\} \cup \{0\} \cup \{1\} = \{-1,0,1\}$.

To find $f^{-1}(B_0)$ we again break the domain into three subsets and

$$f^{-1}(B_0) = (f^{-1}(B_0) \cap (-\infty,0)) \cup (f^{-1}(B_0) \cap \{0\}) \cup (f^{-1}(B_0) \cap (0,-\infty)).$$

We now compute each of these three subsets. (i)

$$f^{-1}(B_0) \cap (-\infty,0) = \{x | x < 0 \text{ and } f(x) = -1 \in B_0\}.$$

Since $-1 \in B_0$ we get: $f^{-1}(B_0) \cap (-\infty,0) = (-\infty,0)$.

(ii)

$$f^{-1}(B_0) \cap \{0\} = \{x \in \mathbb{R} | x = 0, f(x) \in B_0\} = \{0\}$$

since $f(0) = 0 \in B_0$.

(iii)

$$f^{-1}(B_0) \cap (0,\infty) = \{x | x > 0 \text{ and } f(x) = 1 \in B_0\} = \emptyset,$$

since $f(x) = 1 \notin B_0$ for each $x > 0$.

Therefore

$$f^{-1}(B_0) = (-\infty,0) \cup \{0\} \cup \emptyset = (-\infty,0]. \quad \square$$

2. Problem # 8 from the handout. Show that $\mathbb{Z}$ is countable.

Solution. We will show that $\mathbb{Z}$ has the same cardinality as $\mathbb{N}$ by constructing a bijective function $f : \mathbb{Z} \to \mathbb{N}$. Define the function $f : \mathbb{Z} \to \mathbb{N}$ by

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Z},\ x \geq 0 \\ -2x-1 & \text{if } x \in \mathbb{Z},\ x \leq -1 \end{cases}$$

Let us check that this is really a function $\mathbb{Z} \to \mathbb{N}$. If $x \geq 0$ then $f(x) = 2x \in \mathbb{N}$, if $x \leq -1$ then $-2x \geq 2$ and therefore $-2x - 1 \geq 1$. Thus we indeed a function $\mathbb{Z} \to \mathbb{N}$.

We note that $f(\mathbb{N})$ equals the set of even natural numbers $\{0,2,4,...\}$. On the other hand, $f(\{x \in \mathbb{Z} | x \leq -1\})$ is the set of all odd natural numbers $\{1,3,5,...\}$. Therefore $f(\mathbb{Z}) = \mathbb{N}$ and the function f is onto $\mathbb{N}$.

We now check that this function is 1-to-1. The restriction of this function to each subset of the domain $\{x \in \mathbb{Z} | x \leq -1\}$ and $\mathbb{N} = \{x \in \mathbb{Z} | x \geq 0\}$ is injective since $2x = 2y$ implies $x = y$ and $-2x-1 = -2y-1$ implies $x = y$. If f were not injective then this would imply that there are numbers $x \in \mathbb{N}$ and $y \in \{y \in \mathbb{Z} | y \leq -1\} = \mathbb{Z} \backslash \mathbb{N}$ so that $f(x) = f(y)$. However then

$$f(x) = f(y) \in f(\mathbb{N}) \cap f(\mathbb{Z} \backslash \mathbb{N}) = \text{even numbers} \cap \text{odd numbers} = \emptyset.$$

Since $\emptyset$ contains no elements, such x and y cannot exist. $\square$