MATHEMATICS 3210-2. Homework 10. Solutions.

November 8, 2000

1. Problem # 1, page 77 from the textbook. For each of the following prove that there is at least one $x \in \mathbb{R}$ which satisfies the given equation. (You can assume that $\sin(x), \cos(x), e^x$ are continuous on $\mathbb{R}$.) You can also assume that $f(x) = a^x$ is continuous for $0 < a < \infty$, since $a^x = e^{x \log(a)}$.

(a) (5 points). $e^x = x^2$.

Solution. The function $f(x) = e^x - x^2$ is continuous (as the difference of two continuous functions). Note that $f(0) = e^0 - 0 = 1 > 0$. On the other hand, $f(-1) = 1/e - 1 < 0$ since $e > 2 > 1$. Thus, by the intermediate value theorem, there exists $x \in [-1, 0]$ such that $f(x) = 0$. Hence $e^x = x^2$ for this value of x. □

(b) (5 points). $e^x = \cos(x) + 1$.

Solution. The function $f(x) = e^x - \cos(x) - 1$ is continuous (as the difference of two continuous functions). Note that $f(0) = 1 - 1 - 1 = -1 < 0$. On the other hand, $f(\pi) = e^\pi + 1 - 1 = e^\pi > 0$. Thus, by the intermediate value theorem, there exists $x \in [0, \pi]$ such that $f(x) = 0$. Hence $e^x = \cos(x) + 1$ for this value of x. □

(c) (5 points). $2^x = 1 - x$.

Solution. Note that $f(0) = 1 - 1 = 0$, hence $x = 0$ satisfies $2^x = 1 - x$.

To solve this problem using the intermediate value theorem note that the function $f(x) = 2^x - 1 + x$ is continuous (as the difference of two continuous functions). Also, $f(-1) = 1/2 - 1 - 1 = -1.5 < 0$. On the other hand, $f(1) = 2 - 1 - 1 = 2 > 0$. Hence, by the intermediate value theorem, there is a point $x \in [-1, 1]$ such that $f(x) = 0$. Hence $2^x = 1 - x$ for this value of x. □

Another way to solve each of these problems is to notice that

$$\lim_{x \to -\infty} f(x) = L < 0 < \lim_{x \to +\infty} f(x) = M.$$

For instance

$$\lim_{x \to -\infty} 2^x - 1 + x = -\infty, \lim_{x \to \infty} 2^x - 1 + x = +\infty,$$

$$\lim_{x\to-\infty} e^x - x^2 = -\infty, \lim_{x\to+\infty} e^x - x^2 = +\infty.$$

Thus there is $x \in \mathbb{R}$ such that $f(x) = 0$.

2 (10 points). Find a mistake in the following "proof" and find an example of a function which contradicts the "Theorem" below. To justify the example you can use the lower-level calculus, including derivaties.

Theorem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function (this means that if $x > y$ then $f(x) > f(y)$ for any $x, y \in \mathbb{R}$). Then $\lim_{x\to+\infty} f(x) = +\infty$.

"Proof." Given $C \in \mathbb{R}$ we find $K \in \mathbb{R}$ such that $f(K) = C$. Thus for each $x > K$ we have: $f(x) > f(K) = C$. Therefore for each $C \in \mathbb{R}$ there exists K so that for all $x > K$ ($x \in \mathbb{R}$) we have $f(x) > C$. Hence $\lim_{x\to+\infty} f(x) = +\infty$. $\square$

Solution. The mistake is that we implicitely assumed that for each C the number K such that $f(K) = C$ exists. But it may not!

Example. Let $f(x) = -e^{-x}$. Then f is defined for each $x \in \mathbb{R}$, $f'(x) = e^{-x} > 0$ for each x, hence $f(x)$ is strictly increasing: if $x > y$ then $f(x) > f(y)$. However, $f(x) < 0$ for each x, hence

$$\lim_{x\to+\infty} f(x) \neq +\infty.$$

Actually,

$$\lim_{x\to+\infty} -e^{-x} = \lim_{x\to+\infty} e^{-x} = 0.$$

Another example is $\arctan(x) = \tan^{-1}(x) : \mathbb{R} \to (-\pi/2, \pi/2)$, is the inverse to the function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. It is strictly increasing since

$$(\tan^{-1}(x))' = \frac{1}{x^2 + 1} > 0.$$

However $\arctan(x) < \pi/2$ since the arctan is inverse to the function defined on $(-\pi/2, \pi/2)$.