MATHEMATICS 3210-2. Super-bonus problem.

September 19, 2000

The first student to solve this problem will get the A grade for this class no matter what is his/her score on homework, tests, etc. In particular, this student can skip all the tests including the final exam. The deadline for the solution is December 14, 2000.

A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous* at a point $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$. If $f : \mathbb{R} \to \mathbb{R}$ is a function then the *discontinuity set* D of f is the set of points $x \in \mathbb{R}$ at which the function f is not continuous. (That is, f is continuous at each $a \in \mathbb{R}\backslash D$ and is not continuous at each $a \in D$.) For instance, if

$$f(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } -1 < x < 1 \\ -1 & \text{if } x \leq -1 \end{cases}$$

then the discontinuity set of f is $\{-1, 1\}$.

A subset $U \subset \mathbb{R}$ is called *open* if for each $x \in U$ there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U$. For instance, $\mathbb{R}$, $\emptyset$, (a, b) (where $a < b$) are open sets. A subset $Z \subset \mathbb{R}$ is called *closed* if its complement, $\mathbb{R}\backslash Z$, is open. For instance, any finite subset of $\mathbb{R}$ is closed. The sets $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$ are neither open nor closed.

Problem. Show that a subset $C \subset \mathbb{R}$ is the discontinuity set of a function $f : \mathbb{R} \to \mathbb{R}$ $\iff$ C is a union of countably many closed subsets of $\mathbb{R}$ (these subsets are not necessarily disjoint):

$$C = \bigcup_{i=1}^{\infty} C_i, \quad C_i \text{ is closed.}$$

(In particular, if $C \subset \mathbb{R}$ is a countable subset then there is a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous in each point of C and is continuous in each point of $\mathbb{R}\backslash C$.)