Solutions for the 2-nd Midterm Test (Sample).

November 27, 2000

1. For $L, a \in \mathbb{R}$ prove the following:
$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$ if and only if $\lim_{x \to a} f(x) = L$.
Solution. See the textbook.

2. Give the definition of
(a) Limit $\lim_{x \to -\infty} f(x) = L$ (where $L \in \mathbb{R}$).
(b) The derivative of a function f at $a \in \mathbb{R}$.
Solution. See the textbook.

3. Prove that there exists $x \in \mathbb{R}$ such that $\frac{e^x - e^{-x}}{2} = \cos(x)$.

Solution. The function $f(x) = \frac{e^x - e^{-x}}{2} - \cos(x)$ is continuous. $f(0) = 0 - 1 = -1$,

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{2} = \frac{\infty - 0}{2} = \infty.$$

On the other hand, $\cos(x)$ is bounded. Thus there exists $b > 0$ so that $\frac{e^b - e^{-b}}{2} > \cos(b)$. Thus $f(b) > 0$. Hence the function f changes sign on the interval $[0, b]$. Now the intermediate value theorem implies that $f(x) = 0$ for some $x \in (0, b)$. Hence

$$\frac{e^x - e^{-x}}{2} = \cos(x). \quad \square$$

Below is a more concrete way to find b. Consider $b = 2$. Recall that $2 < e$. Thus

$$\frac{e^2 - e^{-2}}{2} \geq \frac{2^2 - 2^{-2}}{2} = \frac{4 - 0.25}{2} > 1.5$$

Since $\cos(b) < 1$, hence $f(b) > 1.5 - 1 = 0.5 > 0$. Thus we can take $b = 2$ and use the same argument as above. $\square$

4. Compute the limit (or show that it does not exist)

$$\lim_{x \to 0+} x \cos(\frac{1}{x}) \log(x).$$

Solution. The function $\cos(\frac{1}{x})$ is bounded. Thus consider

$$\lim_{x\to 0+} x\log(x) = \lim_{x\to 0+} \frac{\log(x)}{1/x}.$$

Both numerator and denominator diverge to $\pm\infty$, hence we can try to use L'Hopital's rule:

$$\lim_{x\to 0+} \frac{(\log(x))'}{(1/x)'} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0.$$

Thus L'Hopital's rule applies and we get:

$$\lim_{x\to 0+} x\log(x) = 0.$$

This means that in

$$\lim_{x\to 0+} \cos(\frac{1}{x}) x\log(x).$$

we have a product of a bounded function by a function which converges to zero as $x \to 0+$. Thus by Limit Theorem for functions,

$$\lim_{x\to 0+} \cos(\frac{1}{x}) x\log(x) = 0. \quad \Box$$

5. (a) Supose that $f : \mathbb{R} \to \mathbb{R}$ is a function which is differentiable everywhere and has $f'(x) > 0$ for each $x \in \mathbb{R}$. Prove that f is strictly increasing.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing differentiable (everywhere) function. Is it true that $f'(x) \geq 0$ for each $x \in \mathbb{R}$?

Solution. (a) See the textbook (Theorem 4.24). (b) The answer is "true". Suppose that $f'(x) < 0$ for some $x \in \mathbb{R}$. Then

$$0 > f'(x) = \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}$$

implies that there exists $\delta > 0$ such that for $0 < h < \delta$ we have:

$$\frac{f(x+h) - f(x)}{h} < 0,$$

which implies that $f(x+h) - f(x) < 0$ and hence $f(x+h) < f(x)$. This contradicts the assumption that f is strictly increasing. $\Box$

Remark. You cannot argue that "if $f'(x) < 0$ then f is strictly decreasing near x", since you do not know that f is countinuously differentiable and hence you do not know that $f'(t) < 0$ for all t near x.