Math 3210-2. Examples of incorrect proofs.

1. Show that the sequence $(-1)^n/n$ does not converge.

"Proof". By the limit theorem 2.12,

$$\lim_{n\to\infty}(-1)^n/n = \lim_{n\to\infty}(-1)^n \cdot \lim_{n\to\infty} 1/n.$$

By Example 2.2, $\lim_{n\to\infty} 1/n = 0$ and by Example 2.3 $\lim_{n\to\infty}(-1)^n$ does not exist. Therefore the product $\lim_{n\to\infty}(-1)^n \cdot \lim_{n\to\infty} 1/n$ does not exist, hence $\lim_{n\to\infty}(-1)^n/n$ does not exist as well. □

The mistake in the proof: In general

$$\lim_{n\to\infty} x_n \cdot y_n \neq \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

The equality holds only provided that both sequences x_n and y_n converge (Theorem 2.12). Thus the very first equality in the proof is incorrect.

Actually, $\lim_{n\to\infty}(-1)^n/n = 0$ (this sequence is the quotient of a bounded sequence by a sequence which diverges to $+\infty$).

Example of a correct argument based on the product of limits:

Compute $\lim_{n\to\infty} \frac{n+1}{n-1} \cdot \frac{(-1)^n}{n^2}$.

Solution.

$$\lim_{n\to\infty} \frac{n+1}{n-1} \cdot \frac{(-1)^n}{n^2} = \lim_{n\to\infty} \frac{n+1}{n-1} \cdot \lim_{n\to\infty} \frac{(-1)^n}{n^2},$$

provided that both limits $\lim_{n\to\infty} \frac{n+1}{n-1}$, $\lim_{n\to\infty} \frac{(-1)^n}{n^2}$ exist. We will show that they exist below, thus justifying this computation.

(i) Computation of $\lim_{n\to\infty} \frac{n+1}{n-1}$:

$$\lim_{n\to\infty} \frac{n+1}{n-1} = \lim_{n\to\infty} \frac{1+1/n}{1-1/n}$$

Since $\lim_{n\to\infty} 1/n = 0$ (Example 2.1), it follows that $\lim_{n\to\infty} 1+1/n = 1, \lim_{n\to\infty} 1 - 1/n = 1 \neq 0$ (by applying Theorem 2.12). Thus, by applying Theorem 2.12 we get:

$$\lim_{n\to\infty} \frac{1+1/n}{1-1/n} = 1/1 = 1.$$

(ii) Computation of $\lim_{n\to\infty} \frac{(-1)^n}{n^2}$.

The sequence $(-1)^n$ is bounded (by -1 from below and by 1 from above). Since $\lim_{n\to\infty} 1/n = 0$, Theorem 2.12 implies that $\lim_{n\to\infty}(1/n)^2 = 0$. Thus Theorem 2.9 implies that $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$.

Therefore we proved that $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$ and that $\lim_{n\to\infty} \frac{n+1}{n-1}$. Hence the existence of both limits (i) and (ii) is justified and

$$\lim_{n\to\infty} \frac{n+1}{n-1} \cdot \frac{(-1)^n}{n^2} = \lim_{n\to\infty} \frac{n+1}{n-1} \cdot \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 1 \cdot 0 = 0. \quad \square$$

If you want to use Theorem 2.12 to prove that limit of product **does not exist** you would have to use arguments of the form:

Theorem. Suppose that (x_n) converges to $a \neq 0$ and (y_n) does not converge. Then the product sequence $(x_n y_n)$ does not converge.

Proof. Suppose to the contrary that $(z_n) = (x_n y_n)$ converges to a real number c. Since $\lim_{n\to\infty} x_n = a \neq 0$, we apply Theorem 2.12 (about quotients) and get:

$$c/a = \lim_{n\to\infty} z_n/x_n = \lim_{n\to\infty} y_n.$$

Thus $\lim_{n\to\infty} y_n = c/a$. Therefore (y_n) converges. However one of the assumptions of theorem was that (y_n) does not converge. Contradiction. Thus the product sequence $(x_n y_n)$ does not converge. □

Think how to apply the same kind of reasoning to sums of sequences, quotients, and so on.

2. Show that

$$\lim_{n\to\infty} \frac{n^2+1}{n} = 0.$$

"Proof". Given $\epsilon > 0$ we need to find $n_0 \in \mathbb{N}$ so that

$$-\epsilon < \frac{n^2+1}{n} < \epsilon$$

for all $n \geq n_0$, $n \in \mathbb{N}$. Let $C = \frac{n^2+1}{\epsilon}$. By the Archimedean Principle, there exists $n_0 \in \mathbb{N}$ so that $n_0 > C$. Therefore for each natural number $n \geq n_0$ we have:

$$n \geq n_0 > C = \frac{n^2+1}{\epsilon} \Rightarrow n > \frac{n^2+1}{\epsilon}$$

By multiplying the last inequality by ϵ and dividing it by n we get the equivalent inequality:

$$\frac{n^2+1}{n} < \epsilon.$$

Note that $\frac{n^2+1}{n} > 0 > -\epsilon$. Therefore for the given $\epsilon > 0$ we found $n_0 \in \mathbb{N}$ so that

$$-\epsilon < \frac{n^2+1}{n} < \epsilon$$

for all $n \geq n_0$, $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} \frac{n^2+1}{n} = 0. \quad \square$$

The mistake in the proof: The Archimedean Principle can be applied only to a fixed number $C \in \mathbb{R}$ (it says that for any real number $C \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ so that $n_0 > C$). Thus to apply A.P. to the number C we have to fix C once and for all. However $C = \frac{n^2+1}{\epsilon}$ depends on n. Therefore we would have to fix the number n as well (say, $n = 100000$). Thus the sentance "for each natural number $n \geq n_0$ we have: $n \geq n_0 > C = \frac{n^2+1}{\epsilon}$" simply does not make sense: we already imposed a condition on n by fixing this number. The "proof" is essentially saying: "for each natural number 100000 which is greater than n_0 we have $100000 \geq n_0 > C = \frac{100000^2+1}{\epsilon}$". However there is only one number 100000 and so the inequality $100000 > \frac{100000^2+1}{\epsilon}$ does not have to hold (consider for instance $\epsilon = 1$). The best you can get here is: for each natural number $m \geq n_0$ we have: $m \geq n_0 > C = \frac{n^2+1}{\epsilon}$. However this does not help you with the limit at all.

The correct solution:

$$\lim_{n\to\infty} n = +\infty, \lim_{n\to\infty} 1/n = 0 \neq -\infty.$$

Therefore by Theorem 2.16,

$$\lim_{n\to\infty} \frac{n^2+1}{n} = \lim_{n\to\infty} (n + 1/n) = +\infty + 0 = +\infty.$$

Thus the sequence $\frac{n^2+1}{n}$ diverges to $+\infty$. □

3. Show that the sequence $x_n = (-1)^n$ does not converge.

"Proof". $x_n = 1$ if n is even and $x_n = -1$ if n is odd.

Thus (x_n) converges to 1 and -1 simultaneously.

However a sequence cannot have two different limits. Thus (x_n) does not converge. □

There is a huge logical gap between the 1-st and the 2-nd sentance of this "proof". Of course, a convergent sequence cannot converge to two different limits. However, how do you deduce that "(x_n) converges to 1 and -1 simultaneously" from "$x_n = 1$ if n is even and $x_n = -1$ if n is odd"?

The correct argument: consider two subsequences (x_{2k}) and (x_{2k+1}) in (x_n) corresponding to the even and the odd indices n. Then

$$\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} 1 = 1,$$

$$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} -1 = -1.$$

Thus the sequence (x_n) contains two subsequences which have different limits. Thus (by Theorem 2.6) the sequence (x_n) does not converge. □

4. Show that $0 \geq 1$.

"Proof." Let $a = 0$, $b = 1$. Then

$$0 \geq 1 \iff a \geq b.$$

Multiplying the both sides of the last inequality by the nonnegative number a we get: $a^2 \geq ab$. However $a^2 = 0^2 = 0$, $ab = 0 \cdot 1 = 0$. Thus

$$a^2 \geq ab \Rightarrow 0 \geq 0.$$

The inequality $0 \geq 0$ is clearly true, thus the initial inequality $0 \geq 1$ is true as well. $\square$

The mistake in the proof: Everything was correct except for the last sentance. What we proved is $0 \geq 1 \Rightarrow 0 \geq 0$. But if an assertion A implies that something is true, it does not follow that A itself is true. In this particular case, "$0 \geq 0$ is true", however it does not imply "$0 \geq 1$ is true. The logical sequence in the "proof" was reversed: you started from something you want to prove and then derived from this a correct assertion. The correct logical sequence should be: start with something you know is true and then arrive to the assertion you want to prove.

To summarize this:

(i). If A is true and $A \Rightarrow B$, then B is true.

(ii). If $A \Rightarrow B$ and B is true, it does not follow that A is true.

(iii) if $A \Rightarrow B$ and B is false, then A is false as well. (This is used to prove things "by contradiction".)

5. Show that the function $f(x) = |x|, f : \mathbb{R} \to \mathbb{R}$ is injective.

"Proof". $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x \leq 0$. Both functions $y = x$ and $y = -x$ are injective. Therefore the restriction of f to $\{x \in \mathbb{R} : x \geq 0\}$ and the restriction of f to $\{x \in \mathbb{R} : x \leq 0\}$ are injective functions. Hence f is injective. $\square$

The mistake in the proof: What was essentially verified:

(a) if $x, x' \geq 0$ and $f(x) = f(x')$ then $x = x'$ (injectivity of the function $y = x$).

(b) if $x, x' \leq 0$ and $f(x) = f(x')$ then $x = x'$ (injectivity of the function $y = -x$).

However the "proof" had omitted one more case: x, x' have the opposite signs, say $x \geq 0, x' \leq 0$. Then $f(x) = x, f(x') = -x'$. To verify injectivity of f you would also have to show that

(c) if $x \geq 0, x' \leq 0$ and $f(x) = f(x')$ then $x = x'$ (which means that $x = x' = 0$).

However if you take $x = 1, x' = -1$ then $f(x) = x = 1, f(x') = -x' = -(-1) = 1$. Thus $f(1) = f(-1)$ and $1 \neq -1$. Therefore f is actually non-injective. $\square$

The moral of this example is that you should check all the possible cases.

6. Show that the sequence $\frac{1}{n}$ does not converge to zero.

"Proof". Suppose $\lim_{n\to\infty} \frac{1}{n} = 0$. This means that for each $\epsilon > 0$, $n \in \mathbb{N} \Rightarrow -\epsilon < 1/n < \epsilon$.

However, take for instance $\epsilon = 1/4$. Then we would get: $-1/4 < 1/n < 1/4$ for all $n \in \mathbb{N}$. But this inequality is clearly false for $n = 2$ since $1/2 > 1/4$. Contradiction. Thus the sequence $\frac{1}{n}$ does not converge to zero. $\square$

The mistake in the proof: The definition of the limit presented here was incorrect. The correct definition requires $-\epsilon < 1/n < \epsilon$ not for all

$n \in \mathbb{N}$ but only for all "sufficiently large" natural numbers n, i.e., all $n \geq n_0$, where $n_0 \in \mathbb{N}$ is some natural number (which will depend on ϵ). For $\epsilon = 1/4$ we would take $n_0 = 5$. Then $1/5 < 1/4$ and therefore for all $n \geq 5$, we have: $-\epsilon < 0 < 1/n < \epsilon = 1/4$.