Math 3210-2. L'Hopital's Theorem.

Theorem 1. *Let $x_0 \in \overline{\mathbb{R}}$, I an open interval containing x_0 or having x_0 as one of its boundary points. Suppose that f, g are differentiable on $I - \{x_0\}$ and $g(x) \neq 0$ and $g'(x) \neq 0$ for each $x \in I - \{x_0\}$. Suppose also that*

$$A = \lim_{x \in I, x \to x_0} f(x) = \lim_{x \in I, x \to x_0} g(x)$$

is either 0 or $\pm\infty$. Then if the limit

$$B = \lim_{x \in I, x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$$

exists then

$$B = \lim_{x \in I, x \to x_0} \frac{f(x)}{g(x)}.$$

Proof. I will prove this theorem only in several special cases. Namely, we shall assume that $I = (a, x_0)$ (i.e., our limits are from the left) and $x_0, B \in \mathbb{R}$.

Case 1. $A = 0$. This is the easiest case. Let $\epsilon > 0$. Start by choosing $\delta > 0$ such that

$$|\frac{f'(c)}{g'(c)} - B| < \epsilon/2$$

for all $c \in (x_0 - \delta, x_0)$ (such δ exists since $B = \lim_{x \to x_0 -} \frac{f'(x)}{g'(x)}$). We will show that for each $x \in (x_0 - \delta, x_0)$ we also have:

$$|\frac{f(x)}{g(x)} - B| < \epsilon.$$

For $x \in (x_0 - \delta, x_0)$ pick a sequence $x_n \in (x, x_0)$ such that $\lim_{n \to \infty} x_n = x_0$. Since $g(x) \neq 0$ and

$$\lim_{n \to \infty} g(x_n) = A = 0,$$

there exists $n_0 \in \mathbb{N}$ such that $g(x) - g(x_n) \neq 0$ for all $n \geq n_0$. Thus we can use the generalized mean value theorem and find (for each $n \geq n_0$) a point $c_n \in (x, x_n)$ such that

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x) - f(x_n)}{g(x) - g(x_n)}.$$

Since $c_n \in (x, x_n) \subset (x_0 - \delta, x_0)$, we also have:

$$|\frac{f(x) - f(x_n)}{g(x) - g(x_n)} - B| = |\frac{f'(c_n)}{g'(c_n)} - B| < \epsilon/2. \quad (2)$$

Recall that $f(x_n) \to 0$ and $g(x_n) \to 0$ as $n \to \infty$. Thus

$$\lim_{n \to \infty} \frac{f(x) - f(x_n)}{g(x) - g(x_n)} = \frac{f(x)}{g(x)}.$$

Therefore there exists $n \in \mathbb{N}$ so that

$$|\frac{f(x)-f(x_n)}{g(x)-g(x_n)} - \frac{f(x)}{g(x)}| < \epsilon/2. \tag{3}$$

We now apply the triangle inequality for the points $\frac{f(x)-f(x_n)}{g(x)-g(x_n)}$, $\frac{f(x)}{g(x)}$, B and get (from the inequalities (2) and (3)):

$$|\frac{f(x)}{g(x)} - B| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, for each $\epsilon > 0$ we found $\delta > 0$ such that

$$|\frac{f(x)}{g(x)} - B| < \epsilon$$

for all $x \in (x_0 - \delta, x_0)$. This proves that

$$B = \lim_{x \to x_0-} \frac{f(x)}{g(x)}. \quad \square$$

Case 2. $A = \infty$. Our arguments in this case are somewhat similar to what we had before. We will need

Lemma 4. *Suppose that $F : (a, b) \to \mathbb{R}$ is a function and $\lim_{x \to b-} F(x) \neq C \in \overline{\mathbb{R}}$. Then there exists a sequence $x_n \to b-$ so that the limit $\lim_{n\to\infty} F(x_n) = L$ exists (here $L \in \overline{\mathbb{R}}$) and $L \neq C$.*

Proof. I will prove this lemma for $C \in \mathbb{R}$ and leave the cases $C = \infty, C = -\infty$ as exercises. Since $\lim_{x\to b-} F(x) \neq C$, there exists $\eta > 0$ so that for each $n \in \mathbb{N}$ we can find $x_n \in (b - 1/n, b)$ such that

$$|F(x_n) - C| \geq \eta.$$

The sequence $F(x_n)$ contains a subsequence $F(x_{n_j})$ which either converges to some $L \in \mathbb{R}$ or diverges to $L = +\infty$ or diverges to $L = -\infty$. Note that in each case $L \notin (C - \eta, C + \eta)$ since $|F(x_n) - C| \geq \eta$. Thus we get $x_{n_j} \to b-$ such that $\lim_{j\to\infty} F(x_{n_j}) = L \neq C$. Renaming this subsequence to (x_n) we get the assertion of lemma. $\square$

Thus (by using the above lemma), if for each sequence $x_n \to x_0-$ such that

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L \in \overline{\mathbb{R}}$$

we also have $B = L$, then $\lim_{x\to x_0-} \frac{f(x)}{g(x)} = L = B$ and we are happy. Hence our goal is to show that $L = B$ for the above sequence (x_n). Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$|\frac{f'(c)}{g'(c)} - B| < \epsilon/2, f(c) > 0,$$

for all $c \in (x_0 - \delta, x_0)$. Since $\lim_{x\to x_0-} f(x) = \infty$, there exists $x \in (x_0 - \delta, x_0)$ so that $f(x) \neq 0$. Similarly, there exists $n_0 \in \mathbb{N}$ such that

$g(x) - g(x_n) \neq 0$ for all $n \geq n_0$. Thus as before we use the generalized mean value theorem and find (for each $n \geq n_0$) a point $c_n \in (x, x_n)$ such that

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x) - f(x_n)}{g(x) - g(x_n)}.$$

Since $c_n \in (x, x_n) \subset (x_0 - \delta, x_0)$, we also have:

$$|\frac{f(x) - f(x_n)}{g(x) - g(x_n)} - B| = |\frac{f'(c_n)}{g'(c_n)} - B| < \epsilon/2. \tag{5}$$

For each nonzero $X, Y \in \mathbb{R}$ we have:

$$\lim_{n\to\infty} \frac{X - f(x_n)}{Y - g(x_n)} = \lim_{n\to\infty} \frac{X/g(x_n) - f(x_n)/g(x_n)}{Y/g(x_n) - 1} = \frac{-L}{-1} = L.$$

(Since $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \infty$.) We apply this to $X = f(x) \neq 0, Y = g(x) \neq 0$. Hence

$$\lim_{n\to\infty} \frac{f(x) - f(x_n)}{g(x) - g(x_n)} = L.$$

I first consider the case $L \in \mathbb{R}$. Then there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ we have:

$$|\frac{f(x) - f(x_n)}{g(x) - g(x_n)} - L| < \epsilon/2. \tag{6}$$

Using the triangle inequality we see that the inequalities (5) and (6) imply

$$|L - B| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since this is true for each $\epsilon > 0$ we get: $L = B$ and we are done.

If $L = +\infty$ (or $L = -\infty$), then

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x) - f(x_n)}{g(x) - g(x_n)} \geq B + \epsilon$$

for all $n \geq n_0$. This contradicts the inequality (5). □