MATH 3100-1.

Proof of the triangle inequality in the plane

Theorem 1. *If A, B, C are arbitrary points in $\mathbb{R}^2$ then $d(A,B)+d(B,C) \leq d(A,C)$.*

Proof. We will identify each point (x,y) with the complex number $z = x+iy$, where $i = \sqrt{-1}$. For the complex number $z = x+iy$ define the absolute value by

$$|z| = \sqrt{x^2+y^2}.$$

Then, if $P = (x,y) = x+iy, Q = (x',y') = x'+iy'$ are points in $\mathbb{R}^2$ then

$$d(P,Q) = \sqrt{(x-x')^2+(y-y')^2} = |P-Q|,$$

because $P - Q = (x-x') + i(y-y')$.

We first will prove a lemma

Lemma 2. *(Triangle inequality for absolute values) If $z = x+iy, z' = x'+iy'$ are complex numbers then*

$$|z+z'| \leq |z| + |z'|.$$

Proof. The above inequality is equivalent to

$$|z+z'|^2 \leq (|z|+|z'|)^2 \iff$$

$$(x+x')^2+(y+y')^2 \leq |z|^2+|z'|^2+2|z||z'| \iff$$

$$x^2+2xx'+(x')^2+y^2+2yy'+(y')^2 \leq x^2y^2+(x')^2+(y')^2+2\sqrt{(x^2+y^2)((x')^2+(y')^2)} \iff$$

$$xx'+yy' \leq \sqrt{(xx')^2+(yy')^2+x^2(y')^2+(x')^2y^2} \iff$$

$$(xx'+yy')^2 \leq (xx')^2+(yy')^2+x^2(y')^2+(x')^2y^2 \iff$$

$$(xx')^2+(yy')^2+2(xx'yy') \leq (xx')^2+(yy')^2+x^2(y')^2+(x')^2y^2 \iff$$

$$0 \leq (xy')^2 - 2(xy')(x'y) + (x'y)^2 \iff$$

$$0 \leq (xy'-x'y)^2.$$

The last inequality is clearly true, hence the first inequality $|z+z'| \leq |z|+|z'|$ is true as well. □

We now can finish the proof of the theorem. Cosndier the points A, B, C. Then

$$d(A,C) = |A-C| = |A-B+(B-C)|.$$

But by the triangle inequality for absolute values we have:

$$|A-B+(B-C)| \leq |A-B| + |(B-C)| = d(A,B)+d(B,C).$$

Therefore $d(A,C) \leq d(A,B)+d(B,C)$. □

Lines in the plane

Consider the 2-dimensional plane $\mathbb{R}^2$. We define straight lines in $\mathbb{R}^2$ as solution sets for equations (in the unknowns x, y):

$$ax + by + c = 0,$$

where a, b, c are arbitrary real numbers so that $(a, b) \neq (0, 0)$ (i.e. we do not allow equations of the form $c = 0$).

Note that translations in the plane move lines to lines. Indeed, if $T(x, y) = (x', y') = (x + p, y + q)$ is a translation and the line L is

$$L = \{(x, y) : ax + by + c = 0\},$$

then $L' = T(L)$ consists of solutions of the equation $a(x' - p) + b(y' - q) + c = 0$, by substituting $x = x' - p, y = y' - q$ into the equation $ax + by + c = 0$.

On the other hand,

$$a(x' - p) + b(y' - q) + c = 0 \iff ax' + by' - ap - bq + c = 0,$$

the latter is again equation of a line.

Exercise 3. *Veryfy that all motions of $\mathbb{R}^2$ send lines to lines. Hint: Check only rotations $R_\alpha(z) = e^{i\alpha}z$ about the origin and the vertical flip.*

Theorem 4. *Given two distinct points P and Q in $\mathbb{R}^2$, there exists a line L through these two points and this line is unique.*

Proof. By applying a translation by $-P$ we can assume that the first point has coordinates $(0, 0)$ (recall that translations send lines to lines).

1. Existence of the line. Let $Q = (x_0, y_0) \neq (0, 0)$. Consider the equation

$$y_0 x - x_0 y = 0.$$

It is clear that $(0, 0)$ and (x_0, y_0) satisfy this equation. Hence the line L given by $y_0 x - x_0 y = 0$ passes through both P and Q.

2. Uniqueness of the line. Suppose that $ax + by + c = 0$ is another line L' through P and Q. Thus $(0, 0)$ is a solution of the equation $ax + by + c = 0$, which implies that $c = 0$. Hence the line L' is given by $ax + by = 0$. Now substitute (x_0, y_0) into this equation. We get:

$$ax_0 + by_0 = 0.$$

Case 1. Suppose that $a \neq 0$. If we would have $y_0 = 0$ then $x_0 = 0/a = 0$, which contradicts the assumption that $Q \neq P$. Thus $y_0 \neq 0$. Hence by solving the equation $ax_0 + by_0 = 0$ we get $b = -a(x_0/y_0)$ and thus the equation for L' takes the form:

$$ax - ax_0/y_0 y = 0.$$

Dividing by a and multiplying by y_0 we get equivalent equation

$$y_0 x - x_0 y = 0,$$

i.e. the equation for L.

Case 2. It is left to consider the exceptional case $a = 0$. Then $b \neq 0$ and we get:

$$0 + by_0 = 0 \iff y_0 = 0,$$

which implies that $x_0 \neq 0$ (recall that $Q \neq P$). Hence we can apply the same argument as in the Case 1, just divide the equation for L' by b. □

Theorem 5. *Given a straight line L there exists a motion m of $\mathbb{R}^2$ so that $m(L)$ is the line $y = 0$.*

Proof Pick a pair of distinct points A, B on L, their distance is $d(A, B) = d > 0$.

By the Homework 10, there exists a motion m so that $m(A) = (0, 0)$, $m(B) = (d, 0)$. Since motion m sends lines to lines, it must send L to the unique line through the points $(0, 0)$ and $(d, 0)$. This line is of course $y = 0$. □

Theorem 6. *(Fifth Postulate) Given a line L and a point P in the plane there exists a unique line L' through P which is parallel to L.*

Proof. By applying a motion of $\mathbb{R}^2$ we can send the line L to the line $y = 0$. Since motions of the plane send parallel lines to parallel lines, we therefore can assume that L is the line $y = 0$.

Suppose that $P = (x_0, y_0)$.

1. Existence of a parallel line. It is clear that the line L' given by $y = y_0$ passes through P and is parallel to L: If $y_0 = 0$ then P belongs to L and $L = L'$, otherwise the lines $y = 0$ and $y = y_0$ do not cross.

2. Uniqueness of a parallel line. Suppose that $ax + by + c = 0$ is a line L'' through P which is parallel to L. Assume for a moment that $a \neq 0$. Then we can substitute $y = 0$ into this equation and get: $x = -c/a$. Thus L'' crosses L at the point $(-c/a, 0)$ and this point of intersection is unique. So, in this case L'' is not parallel to to L.

This forces $a = 0$ and the equation for L'' takes the form $by + c = 0$. Since $a = 0$, we have $b \neq 0$, thus by dividing the equation by b we get:

$$y = -c/b.$$

Recall that (x_0, y_0) is on the line L', thus $-c/b = y_0$ and the equation for L'' takes the form $y = y_0$. However this is the equation of the line L'. Thus L' is unique. □