MATH 3100. 3-rd Midterm Test: Solutions.

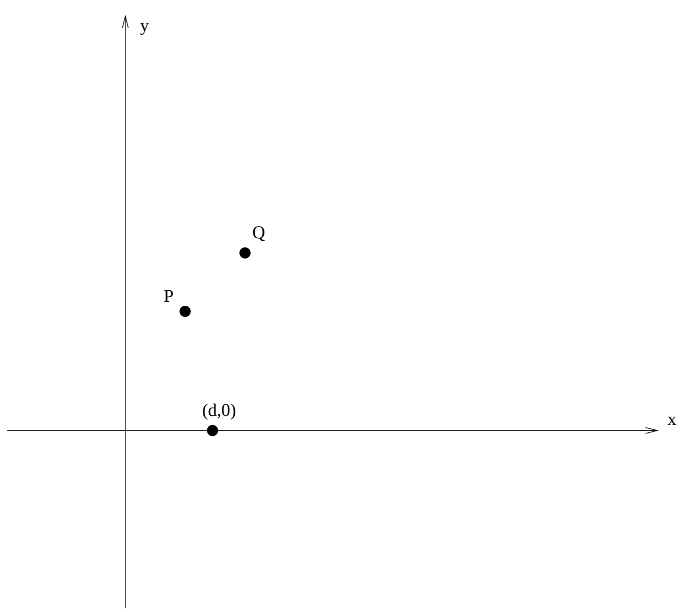
April 17, 2003

1. [20 points] Given points $P = (1,2)$ and $Q = (2,3)$ find a motion m of the plane so that $m((0,0)) = P$ and $m((\sqrt{2},0)) = Q$.

Solution. We first observe that $d = d(P,Q) = \sqrt{(2-1)^2 + (3-2)^2} = \sqrt{2}$. The angle between the line (PQ) and the x-axis equals $\alpha = 45°$. The translation by $(-1,-2)$, $T(z) = z - (1+2i)$ moves the point P to the origin and moves the point Q to the point $(2-1, 3-2) = (1,1)$. Rotation $R_{-\alpha}$ by the angle $-45°$ moves the point $(1,1)$ to the point $(\sqrt{2},0)$. Thus $R_{-\alpha} \circ T(z) = e^{-i\pi/4}(z-(1+2i))$ moves P to $(0,0)$ and Q to $(\sqrt{2},0)$.

We now reverse this process: First apply the rotation R_α and then translation T^{-1} by $(1,2)$: the resulting motion m will send $(0,0)$ to P and $(\sqrt{2},0)$ to Q. Algebraically, this composition is written as

$$m(z) = e^{i\pi/4}z+1+2i = (\cos(\pi/4)+i\sin(\pi/4))z+1+2i = (\sqrt{2}/2+i\sqrt{2}/2)z+1+2i. \quad \square$$



2. [20 points] Using axioms of the plane prove that given a straight line L and a point P in the plane, there exists a straight line L' through the point P which is parallel to L. (You may use without a proof uniqueness of a straight line through the given pair of distinct points, as well as facts about motions of the plane.)

Solution: See the handout. $\square$

3. [20 points] Using the triangle inequality for absolute values of complex numbers prove the triangle inequality for distances between points in the plane.

Alternatively: Prove the triangle inequality for distances between points in the plane (following the textbook).

Solution: See the handout or the textbook. □

4. [20 points] (a) State the definition of a regular n-gon (in the plane).

(b) Suppose that P is a regular n-gon whose sides have length 1. Compute (in terms of n) the radius of the inscribed circle in P. Justify your answer!

Solution: (a) A regular n-gon is a polygon where all n sides are equal and all n angles are equal.

(b) Let's compute the angle $\alpha = \angle AOC$ from the center of the polygon (see Figure): the full angle around the center equals 2π, hence $\alpha = 2\pi/n$. Half of this angle equals π/n. Consider the midpoint B of the side AC of the polygon as on the Figure . Since the triangle ABC is isosceles, we have: OB is the altitude of the triangle ABC and hence $|OB| = r$ is the radius of the inscribed circle. We also have $|AB| = 1/2$, hence

$$|AB|/|OB| = \tan(\alpha), |OB| = |AB|\cot(\alpha) = \frac{\cot(\pi/n)}{2}.$$

Thus

$$r = |OB| = \frac{\cot(\pi/n)}{2}. \quad \square$$

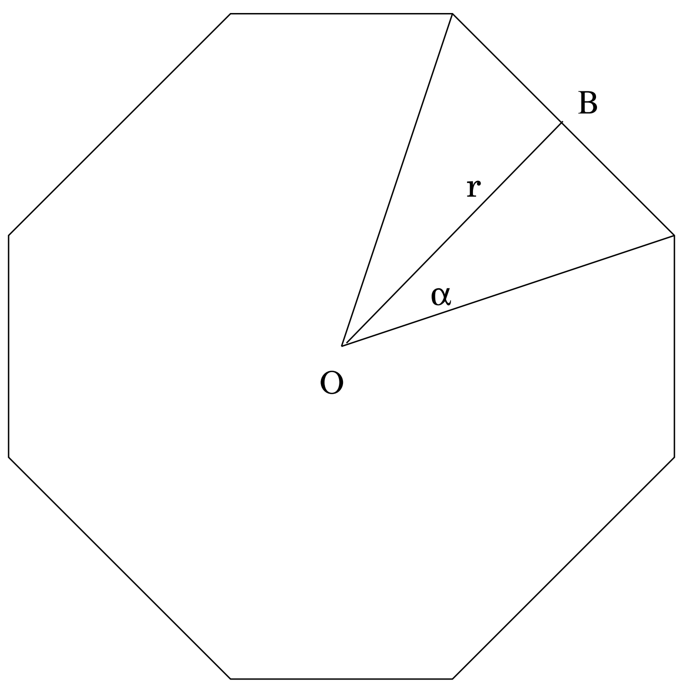


Figure 1:

5. [20 points] State the classification theorem for motions of the plane.

Solution: See the textbook. □