MATHEMATICS 3100-1.

Proof of the concurrence theorems for medians, side bisectors and altitides

Theorem 1. *All medians of the triangle cross in the same point (are concurrent).*

Proof. We first will prove a lemma about the location of the point of intersection of two medians

Lemma 2. *Suppose that BM and CN are medians in a triangle ΔABC. Then their point if intersection D divides each median in the ration $2:1$, i.e. in the Figure 1, $y = |BD| = 2x$, where $x = |DM|$; and $|CD| = 2z$, where $z = |ND|$.*

Proof. I will prove the asseetion for the median BM, since the other one is obtained by relabelling the points on the picture. Extend the medians BM and CN beyond the points M and N distances x and z respectively. We get new points E and F as in the Figure 1. Then the point of intersection M of the diagonals in the quadrilateral $ADCE$ divides each diagonal in half (recall that M is the midpoint of the segment AC). Hence $ADCE$ is a parallelogram. The same argument shows that $AFBD$ is a parallelogram. Since the opposite sides of a parallelogram are congruent, we get that $|AF| = y$. On the other hand, the same two parallelograms show that $AE||FD$ and $FA||DE$ (since opposite sides of a parallelogram are parallel). Hence the quadrilateral $AFDE$ is also a parallelogram. Hence $|FA| = |DE$, which means that $y = 2x$. □

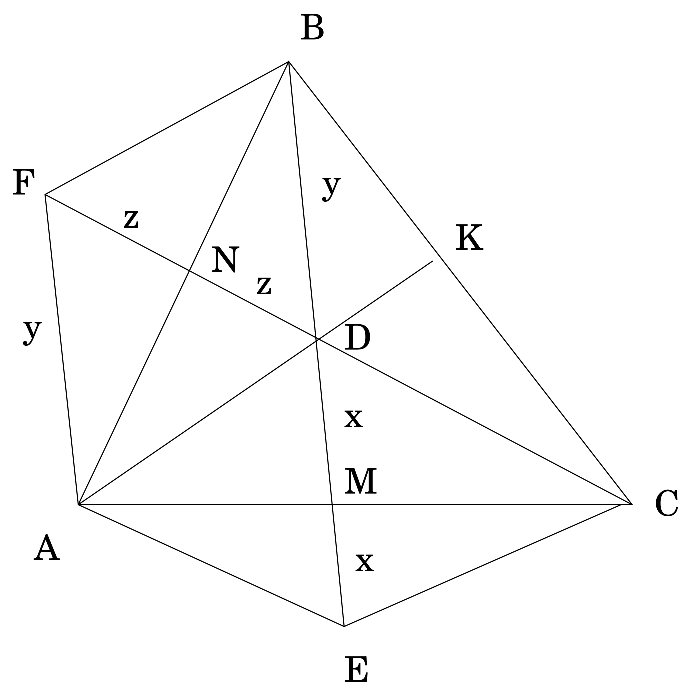

Figure 1: *Medians of the triangle.*

We now can finish the proof of the theorem. Take the point D on the median BM which divides the median in the ratio $2:1$ as above. Then D is the point of intersection of the medians BM and CN. For the same reason, D is the point of intersection of the medians BM and AK. Hence all medians cross in the same point. □

Through the midpoint of each side of a triangle ΔABC draw a perpendicular line: this line is called the *perpendicular bisector* of this side.

Theorem 3. *All three perpendicular bisectors cross at the same point. This point is the center of the circle passing through the vectices of the triangle. (Called "circumscribed circle" of the triangle.)*

Proof. Perpendicular bisector of a segment XY consists of points which are within the same distance from X as from Y. Hence, if O is the point of intersection between L (bisector of AB) and L' (bisector of AC) then

$$|OA| = |OB|, |OA| = |OC|,$$

hence $|OB| = |OC|$. Thus O lies on the perpendicular bisector of BC. Hence all three perpendicular bisectors cross in the same point O. N

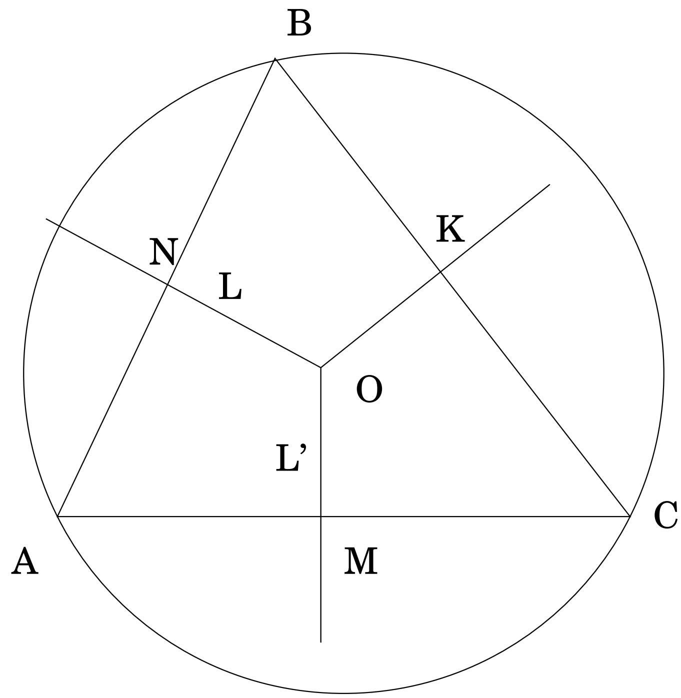

Figure 2: *Perpendicular bisectors and the circumscribed circle.*

Finally, note that since $|OA| = |OB| = |OC| = R$, the circle with center at O and radius R passes through all three vertices of the triangle. □

Theorem 4. *All three altitudes of a triangle ΔABC cross at the same point.*

Proof. Through each vertex of the triangle ΔABC darw a line parallel to the opposite side of the triangle. We thus get a new triangle ΔPQS. Observe that the points A, B, C are the midpoints of the sides of ΔPQS. Indeed: the quadrilateral $PBCA$ is a parallelogram, hence $|PB| = |AC|$; since $ABQC$ is also a parallelogram, $|AC| = |BQ|$, hence B is a midpoint. The same works for the other vertices of ΔABC.

The altitude BZ of ΔABC is perpendicular to AC, hence it is also perpendicular to the line PQ 9which is parallel to AC). hence the altitide BZ is the perpendicular bisector of PQ. The same applies to the two other altitides: they are perpendicular bisectors of the triangle ΔPQS. Since perpendicular bisectors of a triangle ΔPQS cross in the same point, the altitudes of ΔABC cross in the same point (the center of the circumscribed circle of ΔPQS). $\square$

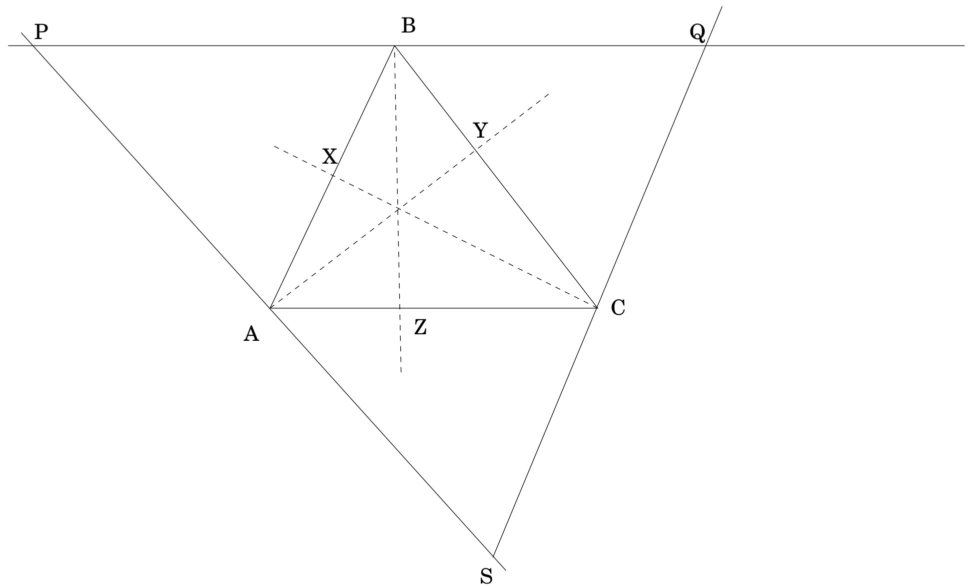

Figure 3: *Altitudes and perpendicular bisectors.*

Consider a triangle ΔABC in Figure 4. A *bisector* of the angle BAC of this triangle is the line through A which divides the angle in half. Bisectors of the other angles are defined in the same fashion.

Theorem 5. *All three bisectors of a triangle ΔABC cross at the same point o. This point is the center of the* **inscribed** *circle of the triangle ΔABC.*

Proof. The bisector of the angle BAC is the line consisting of points P which are within the same distance from the sides AB and AC. To check this consider the perpendiculars from P to the sides AB and AC. The triangles ΔAQP and ΔATP are congruent $\iff$ $\angle QAT = \angle TAQ$, or equivalently, $|PQ| = |PT|$.

Hence, if o is the point of intersection between the bisectors of the angles BAC and ABC, then o is within equal distance r from all three sides of the triangle ΔABC. Hence o also belongs to the bisector of the angle BCA. Hence all three bisectors cross at the same point, which is within the same distance r from all three sides of ΔABC.

Draw the perpendiculars oX, oY, oZThe points X, Y, Z are within distance r from o. Therefore the circle with the center o and radius r is tangent to ΔABC. $\square$

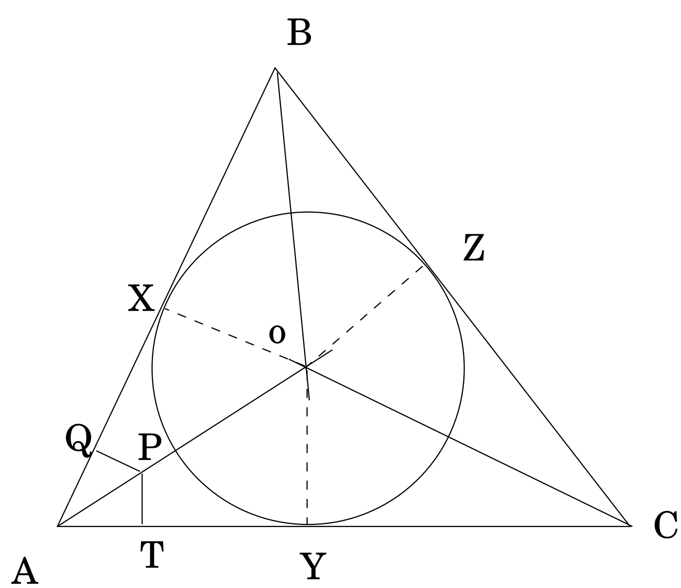

Figure 4: *Bisectors and the inscribed circle.*