MATHEMATICS 3220-1. Third Midterm Test (Sample): Solutions.

April 28, 2002

1. [15 points] State the theorem which guarantees differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of the partial derivatives $\frac{\partial f_i}{\partial x_j}$.

Solution. See theorem 11.15 in the textbook.

2. [15 points] Write Taylor's formula for the function $f(x, y) = x^3 y$ for $p = 3$ at $\mathbf{a} = (0, 1)$.

Solution. Taylor's formula for $p = 3$ is:

$$f(x, y) = f(0, 1) + \sum_{k=1}^{2} \frac{1}{k!} D^{(k)} f((0, 1), \mathbf{h}) + \frac{1}{3!} D^{(3)} f(\mathbf{c}; \mathbf{h})$$

where $\mathbf{x} = (x, y) = (0, 1) + (h_1, h_2)$, $\mathbf{h} = (h_1, h_2)$ and $\mathbf{c} = (c, d)$ is a point on the segment $[\mathbf{a}, \mathbf{x}]$. The partial derivatives of f are:

$$f_x = 3x^2 y,\ f_y = x^3,$$

$$f_{xx} = 6xy,\ f_{xy} = 0,\ f_{yy} = 0,$$

$$f_{xxx} = 6y,\ f_{xxy} = 6x,\ f_{xyy} = 0,\ f_{yyy} = 0.$$

Thus Taylor's formula is

$$f(h_1, 1 + h_2) = 0 + \frac{1}{3!} D^{(3)} f(\mathbf{c}; \mathbf{h}) = \frac{1}{6}[6dh_1^3 + 18ch_1^2 h_2] = dh_1^3 + 3ch_1^2 h_2.$$

□

3. [20 points] State the definition of a convex set and prove that the set

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

is convex.

Solution. A subset $E \subset \mathbb{R}^n$ is called convex if for each pair of points $\mathbf{p}, \mathbf{q}$ in E the straight-line segment $[\mathbf{p}, \mathbf{q}]$ between $\mathbf{p}$ and $\mathbf{q}$, is also contained in E.

Let's now prove that the set E in the problem is convex. Let $\mathbf{p}, \mathbf{q} \in E$, thus $\|\mathbf{p}\| \leq 1, \|\mathbf{q}\| \leq 1$. The points in the segment $[\mathbf{p}, \mathbf{q}]$ have the form $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$, where $t \in [0,1]$. Thus we have to verify that $\|x\| \leq 1$ for every $t \in [0,1]$. By the triangle inequality we have:

$$\|\mathbf{x}\| = \|(1-t)\mathbf{p} + t\mathbf{q}\| \leq \|(1-t)\mathbf{p}\| + \|t\mathbf{q}\| = (1-t)\|\mathbf{p}\| + t\|\mathbf{q}\| \leq (1-t) + t = 1. \quad \square$$

4. [15 points] Suppose that $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable everywhere, $f(0) = (0,0)$, $f'(0) = (1,2)$ and $\nabla g(0,0) = (2,3)$. Let $h = g \circ f$. Compute $h'(0)$.

Solution. Since f and g are differentiable, by the Chain Rule the function $h = g \circ f$ is also differentiable and

$$h'(0) = Dh(0) = Dg(f(0))Df(0) = \nabla g(0) \cdot f'(0) = (2,3) \cdot (1,2) = 2 + 6 = 8.$$

$\square$

5. [15 points] For the following function prove that f^{-1} exists and is differentiable in some nonempty open set containing $\mathbf{a}$ and compute $D(f^{-1})(\mathbf{a})$:

$$f(u,v) = (3u - v, 2u + 5v), \mathbf{a} = (1,1).$$

Solution. First, let's find a point $\mathbf{p} \in \mathbb{R}^2$ such that $f(\mathbf{p}) = \mathbf{a}$. The point $\mathbf{p} = (u,v)$ satisfies:

$$\begin{cases} 3u - v = 1 \\ 2u + 5v = 1 \end{cases}$$

Solving this system we get: $u = \frac{6}{17}, v = \frac{1}{17}$. Hence

$$\mathbf{p} = (\frac{6}{17}, \frac{1}{17}).$$

Let's compute $Df(\mathbf{p})$. Since f is linear,

$$A = Df(\mathbf{p}) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}.$$

The determinant of this matrix is $17 \neq 0$. Thus we can apply the inverse function theorem, accoriding to which f^{-1} exists and is differentiable in some nonempty open set containing $\mathbf{a}$ and

$$D(f^{-1})(\mathbf{a}) = A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}.$$

6. [20 points] State and prove the theorem about differentiability and the total derivative of the sum of two vector-valued functions. (You can use without a proof the basic lemma with the equivalent definition of differentiability.)

Solution. See theorem 11.22 and your class notes. $\square$