MATHEMATICS 3210-2. First Midterm Test (Sample): Solutions.

January 25, 2002

1. [15 points] Using the definition of the limit of a sequence prove that the following sequence converges:

$$\mathbf{x}_k = (\frac{k}{k-1}, \frac{1}{k^2}).$$

Solution. I claim that the limit of this sequence is $(1, 0)$. Let $\epsilon > 0$, we have to find K such that for all $k > K$ we have:

$$\|(\frac{k}{k-1} - 1, \frac{1}{k^2})\| < \epsilon.$$

Since $\|v\| \leq \sqrt{2}\|v\|_\infty$, it suffices to find K_1, K_2 such that for all $k > K = \max(K_1, K_2)$ we have:

$$|\frac{k}{k-1} - 1| < \epsilon/\sqrt{2}, \frac{1}{k^2} < \epsilon/\sqrt{2}.$$

To find K_1, K_2 we "solve" each of the above inequalities:

$$|\frac{k}{k-1} - 1| < \epsilon/\sqrt{2} \iff \frac{1}{k-1} < \epsilon/\sqrt{2}$$

(assuming $k > 1$), equivalently, $\frac{\sqrt{2}}{\epsilon} < k - 1 \iff$

$$\frac{\sqrt{2}}{\epsilon} - 1 < k.$$

Hence we can take $K_1 = \max(2, \frac{\sqrt{2}}{\epsilon} - 1)$.

For the second inequality we have:

$$\frac{1}{k^2} < \epsilon/\sqrt{2} \iff \frac{\sqrt{2}}{\epsilon} < k^2 \iff$$

$$\frac{2^{1/4}}{\sqrt{\epsilon}} < k.$$

Hence we take $K_2 := \frac{2^{1/4}}{\sqrt{\epsilon}}$. Then for each $k \geq \max(K_1, K_2)$ we have both

$$|\frac{k}{k-1} - 1| < \epsilon/\sqrt{2}, \frac{1}{k^2} < \epsilon/\sqrt{2}. \quad \square$$

2. [15 points] State the Bolzano-Weierstrass theorem for $\mathbb{R}^n$.
See the textbook.

3. [20 points] Compute the following limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2}$$

(you can use limit theorems).

Solution. Note that $\lim_{(x,y)\to(0,0)} x^2 = 0$. We will verify that the function $\frac{x^2}{x^2+y^2}$ is bounded. Note that $x^2 \le x^2+y^2$, hence $\frac{x^2}{x^2+y^2} \le 1$. Thus

$$0 \le \frac{x^4}{x^2+y^2} \le x^2.$$

Since the function x^2 is continuous, $\lim_{(x,y)\to(0,0)} x^2 = 0$, thus by the squeeze theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} = 0.$$

4. [15 points] Let $f(x,y) = (x, y^2)$. Using the definition of total derivative verify that

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 2y \end{bmatrix}.$$

Solution. Let $\mathbf{a} = (a,b) \in \mathbb{R}^2$. Let

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 2b \end{bmatrix}$$

Then we have to verify:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T\mathbf{h}\|}{\|\mathbf{h}\|} = 0.$$

The vector in the numerator is:

$$\begin{bmatrix} a+x \\ (b+y)^2 \end{bmatrix} - \begin{bmatrix} a \\ b^2 \end{bmatrix} - \begin{bmatrix} x \\ 2by \end{bmatrix} = \begin{bmatrix} 0 \\ y^2 \end{bmatrix}.$$

The norm of this vector is y^2. The norm of the vector in the denominator is

$$\|\mathbf{h}\| = \sqrt{x^2+y^2}.$$

Hence we have to show that

$$\lim_{x\to 0, y\to 0} \frac{y^2}{\sqrt{x^2+y^2}} = 0.$$

Note that

$$\lim_{x\to 0, y\to 0} \frac{y^4}{x^2+y^2} = 0$$

(see problem 3). Hence, since the square root is a continuous function,

$$\lim_{x\to 0, y\to 0} \frac{y^2}{\sqrt{x^2+y^2}} = \lim_{x\to 0, y\to 0} \sqrt{\frac{y^4}{x^2+y^2}} = 0. \quad \square$$

5. [15 points] Determine if the subset $\{(x,y) : x = 0, y \in \mathbb{R}\}$ of $\mathbb{R}^2$ is open. Give a proof!

Solution. I claim that this set A is not open. Take $p = (0,0) \in A$. For each $\epsilon > 0$ the open ball $B_\epsilon(p)$ will contain the point

$$(\epsilon/2, 0)$$

which does not belong to A. □

6. [20 points] Reorder the following sentances to get a valid proof of the theorem on uniqueness of limit of a function:

Proof. By the triangle inequality we get:

$$\|\mathbf{v}-\mathbf{w}\| \leq \|f(\mathbf{x})-\mathbf{v}\| + \|f(\mathbf{x})-\mathbf{w}\| < 2\epsilon = \|\mathbf{v}-\mathbf{w}\|.$$

Suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{v}, \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{w}$. Suppose that $\mathbf{v} \neq \mathbf{w}$, then $\|\mathbf{v}-\mathbf{w}\| > 0$. Thus $\|\mathbf{v}-\mathbf{w}\| < \|\mathbf{v}-\mathbf{w}\|$. Contradiction. Hence for all $\mathbf{x}$ satisfying $\|\mathbf{x}-\mathbf{a}\| < \min(\delta_1, \delta_2)$ we have:

$$\|f(\mathbf{x})-\mathbf{v}\| < \epsilon, \|f(\mathbf{x})-\mathbf{w}\| < \epsilon.$$

Let $\epsilon = \|\mathbf{v}-\mathbf{w}\|/2$. Then (since $\lim_{x\to a} f(x) = \mathbf{v}$), there exists $\delta_1 > 0$ such that $0 < \|\mathbf{x}-\mathbf{a}\| < \delta_1$ implies $\|f(\mathbf{x})-\mathbf{v}\| < \delta_1$. Similarly, (since $\lim_{x\to a} f(x) = \mathbf{w}$), there exists $\delta_2 > 0$ such that $0 < \|\mathbf{x}-\mathbf{a}\| < \delta_2$ implies $\|f(\mathbf{x})-\mathbf{w}\| < \delta_2$. □

Solution. Suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{v}, \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{w}$. Suppose that $\mathbf{v} \neq \mathbf{w}$, then $\|\mathbf{v}-\mathbf{w}\| > 0$. Let $\epsilon = \|\mathbf{v}-\mathbf{w}\|/2$. Then (since $\lim_{x\to a} f(x) = \mathbf{v}$), there exists $\delta_1 > 0$ such that $0 < \|\mathbf{x}-\mathbf{a}\| < \delta_1$ implies $\|f(\mathbf{x})-\mathbf{v}\| < \delta_1$. Similarly, (since $\lim_{x\to a} f(x) = \mathbf{w}$), there exists $\delta_2 > 0$ such that $0 < \|\mathbf{x}-\mathbf{a}\| < \delta_2$ implies $\|f(\mathbf{x})-\mathbf{w}\| < \delta_2$. Hence for all $\mathbf{x}$ satisfying $\|\mathbf{x}-\mathbf{a}\| < \min(\delta_1, \delta_2)$ we have:

$$\|f(\mathbf{x})-\mathbf{v}\| < \epsilon, \|f(\mathbf{x})-\mathbf{w}\| < \epsilon.$$

By the triangle inequality we get:

$$\|\mathbf{v}-\mathbf{w}\| \leq \|f(\mathbf{x})-\mathbf{v}\| + \|f(\mathbf{x})-\mathbf{w}\| < 2\epsilon = \|\mathbf{v}-\mathbf{w}\|.$$

Thus $\|\mathbf{v}-\mathbf{w}\| < \|\mathbf{v}-\mathbf{w}\|$. Contradiction. □