Math 3220-1. Final Test: Solutions.

1. [15 points] Using the definition of the limit of a sequence prove that the following sequence converges:

$$\mathbf{x}_k = (\frac{k}{k+1}, \frac{(-1)^k}{k^2}).$$

Solution. We will show that the limit of this sequence equals $(1, 0)$, in other words, for each $\epsilon > 0$ we want to find $n_0 \in \mathbb{N}$ such that for all $k \geq n_0 + 1$,

$$\epsilon > \|(\frac{k}{k+1} - 1, \frac{(-1)^k}{k^2})\| = \|(\frac{-1}{k+1}, \frac{(-1)^k}{k^2})\|.$$

Since $\|\mathbf{x}\| \leq \sqrt{2}\|\mathbf{x}\|_\infty$, it suffices to find n_0 so that for all $k \geq n_0$,

$$\frac{1}{k+1} < \epsilon/\sqrt{2}, \frac{1}{k^2} < \epsilon/\sqrt{2}. \tag{1}$$

Let's solve both inequalities for k:

$$\frac{1}{k+1} < \epsilon/\sqrt{2} \iff \frac{\sqrt{2}}{\epsilon} < k+1,$$

hence for the first inequality it suffices to take $k \geq n_1 = [\frac{\sqrt{2}}{\epsilon}]$.

We also have

$$\frac{1}{k^2} < \epsilon/\sqrt{2} \iff \frac{\sqrt{2}}{\epsilon} < k^2 \iff \frac{2^{1/4}}{\sqrt{\epsilon}} < k.$$

Hence for the second inequality it suffices to take $k \geq n_2 = [\frac{2^{1/4}}{\sqrt{\epsilon}}]$.

Hence, by taking $n_0 = \max([\frac{2^{1/4}}{\sqrt{\epsilon}}], [\frac{\sqrt{2}}{\epsilon}])$ we get the required assertion.

2. [15 points] State and prove theorem about equivalence of convergence of a sequence $\mathbf{x}_k$ in $\mathbb{R}^n$ and convergence of the coordinate sequences $x_k(j)$, $j = 1, ..., n$. (You can use relation between norm and the sup-norm.)

Solution. See the textbook.

3. [10 points] Compute the iterated limits of

$$f(x, y) = \frac{\sin(x)\sin(y)}{x^2 + y^2}.$$

Decide if the function has a limit as $(x, y) \to (0, 0)$ and prove that the limit exists (or does not exist).

Solution. The iterated limits are:

$$\lim_{y\to 0}\lim_{x\to 0} \frac{\sin(x)\sin(y)}{x^2 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0,$$

$$\lim_{x\to 0}\lim_{y\to 0} \frac{\sin(x)\sin(y)}{x^2 + y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Hence both iterated limits are equal to zero. Now consider the limit along the line $x = y$:

$$\lim_{x\to 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2}\lim_{x\to 0} \frac{\sin^2(x)}{x^2}.$$

Recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (for instance, from the l'Hospital's rule). Hence by the product theorem for the limits we get:

$$1 = \lim_{x\to 0}(\frac{\sin(x)}{x})^2 = \lim_{x\to 0} \frac{\sin^2(x)}{x^2}.$$

Thus

$$\lim_{x\to 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2} \neq 0.$$

Hence the limit along the line $x = y$ is different from the iterated limits, so the limit of the function does not exist. □

4. [10 points] State the Bolzano-Weierstrass theorem for $\mathbb{R}^n$.
Solution. See the textbook.

5. [15 points] Find the interior of the set

$$E = \{(x, y) \in \mathbb{R}^2 : xy \leq 0\}.$$

(You can use theorems about continuous functions here.)

Solution. Let's show that the set $A = \{(x, y) : xy > 1\}$ is open. Indeed, this set is given by a strict inequality with continuous left hand side, so the set A is open. I claim that $A = E^0$. To prove this I have to verify that each point $\mathbf{p} = (x, y)$ such that $xy = 1$ is a "bad" point of E, i.e. there is a sequence $\mathbf{x}_k \in E^c$ such that $\lim_{k\to\infty} \mathbf{p}_k = \mathbf{p}$. Let $\mathbf{p}_k = (x_k, y_k) = (1 - \frac{1}{k})(x, y)$. Then

$$\lim_{k\to\infty} \mathbf{p}_k = \lim_{k\to\infty} (1 - \frac{1}{k})(x, y) = (x, y).$$

On the other hand, $x_k y_k = (1 - \frac{1}{k})^2 < 1$ for each $k \in \mathbb{N}$. Hence $\mathbf{p}_k \notin E$ for all k.

6. [10 points] Determine if the set $E = \mathbb{R}^2 \setminus \{(x, y) : y \neq x^2\}$ is connected.

Solution. I claim that the set E is not connected. Let $U = \{(x, y) : y > x^2\}$, $V = \{(x, y) : y < x^2\}$. Then $U \cap V = E$, both U and V are open (since they are given by strict inequalities with continuous left hand side) and they are both nonempty. Hence $\{U, V\}$ is a separation of E, so E is not connected.

7. [15 points] Suppose that $V \subset \mathbb{R}^n$, $f : V \to \mathbb{R}^2$ is continuous and the image of f is the set $E = \{(x, y) : xy = 1\}$. Determine if the set V is compact. (If you claim any property about E you should prove this property.)

Solution. First, note that E contains points $\mathbf{p}_k = (k, \frac{1}{k})$, which form an unbounded sequence. Thus E is not bounded, hence E is not compact. Recall that image of a compact set under a continuous function is again compact. Since V is not compact, E is noncompact as well. □

8. [10 points] Which of the following are true and which are false (you do not have to give a proof) for a function $f : \mathbb{R}^n \to \mathbb{R}^m$:

(a) If f is differentiable then f is continuous.

(b) If f is not differentiable at some $\mathbf{a} \in \mathbb{R}^n$ then some partial derivatives $\frac{\partial f_i}{\partial x_j}$ do not exist.

(c) If for some i, j, the partial derivative $\frac{\partial f_i}{\partial x_j}$ exists but is not continuous then f is not differentiable.

Solution. (a) is true. (b) and (c) are false. □