Math 3220-1. 3-rdd Midterm Test: Solutions.

1. [15 points] Write Taylor's formula for the function $f(x,y) = \sin(x) + \cos(y)$ for $p = 3$ at $\mathbf{a} = (0,0)$.

Solution. Taylor's formula for $p = 3$ is:

$$f(x,y) = f(0,0) + \sum_{k=1}^{2} \frac{1}{k!} D^{(k)} f((0,0), \mathbf{h}) + \frac{1}{3!} D^{(3)} f(\mathbf{c}; \mathbf{h})$$

where $\mathbf{x} = (x,y) = (0,0) + (h_1, h_2)$, $\mathbf{h} = (h_1, h_2)$ and $\mathbf{c} = (c,d) = (th_1, th_2)$ is a point on the segment $[\mathbf{a}, \mathbf{x}]$ (i.e. $0 \leq t \leq 1$). The partial derivatives of f are:

$$f_x = \cos(x), f_y = -\sin(y),$$

$$f_{xx} = -\sin(x), f_{xy} = 0, f_{yy} = -\cos(x),$$

$$f_{xxx} = -\cos(x), f_{xxy} = 0, f_{xyy} = 0, f_{yyy} = \sin(y).$$

Thus Taylor's formula is

$$f(h_1, h_2) = 1 + h_1 + \frac{1}{2}(-h_2^2) + \frac{1}{3!} D^{(3)} f(\mathbf{c}; \mathbf{h}) = h_1 - \frac{h_2^2}{2} + \frac{1}{6}[-\cos(c) h_1^3 + \sin(d) h_2^3];$$

$$f(h_1, h_2) = 1 + h_1 - \frac{h_2^2}{2} + \frac{1}{6}[-\cos(th_1) h_1^3 + \sin(th_2) h_2^3]$$

for some $t \in [0,1]$. □

2. [15 points] State the inverse function theorem.

Solution. WARNING: The inverse function theorem is stated somewhat incorrectly in the textbook, namely the "onto" part is missing. The correct statement is:

Inverse Function Theorem. Suppose that $V \subset \mathbb{R}^n$ is an open subset, $f : V \to \mathbb{R}^n$ is a continuously differentiable function. Then for each $\mathbf{a} \in V$ for which $\Delta_f(\mathbf{a}) \neq 0$ there exists an open subset $W \subset V$ so that:

1. The restriction $f|_W$ of f to W is 1-1 and $f(W) = U$ is open in $\mathbb{R}^n$. (The second part of this sentance was missing in the textbook).
2. The inverse function $f^{-1} : U \to W$ is continuously differentiable.
3. For each $\mathbf{x} \in W$ and $\mathbf{y} = f(\mathbf{x})$ we have:

$$Df^{-1}(\mathbf{y}) = [Df(\mathbf{x})]^{-1}.$$

Note that without the 2-nd part of (1), the part (2) does not make sense since we did not define continuously differentiable functions whose domain is not open.

□

3. [20 points] State and prove the theorem about differentiability and the total derivative of the sum of two vector-valued functions. (You can use without a proof the basic lemma with the equivalent definition of differentiability.)

Solution. See the textbook. □

4. [20 points] Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$.

Solution. If f were differentiable at $(0,0)$ then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{Df}(\mathbf{0})(\mathbf{h})}{\|\mathbf{h}\|} = 0.$$

First, let's compute $f_x(0,0)$ and $f_y(0,0)$:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} 0 = 0,$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} 0 = 0.$$

Assume now that f is differentiable at $(0,0)$, then $Df(0,0) = (0,0)$, hence

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{h})}{\|\mathbf{h}\|} = 0.$$

We would get the same limit by considering $\mathbf{h} = (t,t)$ where $t > 0$. However

$$\lim_{t\to\mathbf{0}+} \frac{f(t,t)}{t\sqrt{2}} = \lim_{t\to\mathbf{0}} \frac{t}{t\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Since $\frac{1}{\sqrt{2}} \neq 0$ we get a contradiction. □

5. [15 points] Suppose that V is a convex open subset in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in V so that $Df(\mathbf{x})$ is zero for each $\mathbf{x} \in V$. Using the Mean Value Theorem show that f is constant in V.

Solution. Let $\mathbf{a} \in V$. By the Mean Value Theorem, for each $\mathbf{x} \in V$ there exists a point $\mathbf{c} \in [\mathbf{a}, \mathbf{x}]$ such that

$$1 \cdot (f(\mathbf{x}) - f(\mathbf{a})) = 1 \cdot Df(\mathbf{c})\mathbf{h},$$

where $\mathbf{h} = \mathbf{x} - \mathbf{a}$. Since $Df(\mathbf{c}) = 0$, the right hand side of the above equation is zero. Hence $f(\mathbf{x}) = f(\mathbf{a})$ for each $\mathbf{x} \in V$. Thus f is constant, equal to $f(\mathbf{a})$. □

6. [15 points] Suppose that $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable everywhere, $f(0) = (0,0)$, $f'(0) = (2,1)$ and $\nabla g(0,0) = (1,3)$. Let $h = g \circ f$. Compute $h'(0)$.

Solution. Since f and g are differentiable, by the Chain Rule the function $h = g \circ f$ is also differentiable and

$$h'(0) = Dh(0) = Dg(f(0))Df(0) = \nabla g(0) \cdot f'(0) = (1,3) \cdot (2,1) = 2 + 3 = 5.$$

□