Math 3220-1. 1-st Midterm Test: Solutions.

1. [15 points] State the definitions of open sets and closed sets in $\mathbb{R}^n$. Give an example of a set which is both open and closed.

Solution. See the textbook for the definitions. The example is $\mathbb{R}^n$ or $\emptyset$.

2. [15 points] State and prove theorem about equivalence of convergence of a sequence $\mathbf{x}_k$ in $\mathbb{R}^n$ and convergence of coordinate sequences $x_k(j)$, $j = 1, ..., n$.

Solution. See the textbook.

3. [20 points] Compute the limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(\frac{1}{x})\sqrt{x}y^2}{x^2+y^2}$$

You can use limit theorems if you like.

Solution. Note that the function $|\sin(\frac{1}{x})|$ is bounded by 1,

$$\lim_{(x,y)\to(0,0)} \sqrt{x} = \lim_{x\to 0} \sqrt{x} = 0,$$

since $\sqrt{x}$ is continuous. It remains to note that $y^2 \leq x^2 + y^2$ implies that

$$\frac{y^2}{x^2+y^2} \leq 1.$$

Hence $\frac{\sin(\frac{1}{x})\sqrt{x}y^2}{x^2+y^2}$ is the product of a bounded function $\frac{\sin(\frac{1}{x})y^2}{x^2+y^2}$ by the function $\sqrt{x}$ whose limit is zero. Hence

$$\lim_{(x,y)\to(0,0)} \frac{\sin(\frac{1}{x})\sqrt{x}y^2}{x^2+y^2} = 0.$$

4. [20 points] Using definition of total derivative prove that the function

$$f(x, y) = (x - y, x^2 + y^2)$$

is differentiable on $\mathbb{R}^2$ and its total derivative equals

$$Df(x, y) = \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix}.$$

You can use limit theorems if you like.

Solution. Let $\mathbf{a} = (a, b) \in \mathbb{R}^2$, then we have to verify that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

Let (x, y) denote the vector $\mathbf{h}$. First, consider the vectors in the numerator:

$$= \begin{bmatrix} a + x - (b + y) \\ (a + x)^2 + (b + y)^2 \end{bmatrix} - \begin{bmatrix} a - b \\ a^2 + b^2 \end{bmatrix} - \begin{bmatrix} x - y \\ 2ax + 2by \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ x^2 + y^2 \end{bmatrix}$$

The norm of this vector is x^2+y^2. The norm of the vector $\mathbf{h}$ is $\sqrt{x^2+y^2}$. Hence, we have to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0.$$

Note that $\frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2$, is a polynomial function; hence, as we discussed in the class, this function is continuous. Thus the function $\sqrt{x^2+y^2}$, which is the composition of x^2+y^2 with continuous function $\sqrt{t}$, is continuous as well. Hence

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0. \quad \square$$

5. [15 points] Prove that the sequence

$$\mathbf{x}_k = ((-1)^k + \frac{1}{k}, \sin(k)\cos(k^2))$$

in $\mathbb{R}^2$ contains a convergent subsequence. You can use limit theorems if you like.

Solution. Note that $|\sin(k)\cos(k^2)| \leq 1$ and by the triangle inequality $|(-1)^k + \frac{1}{k}| \leq |(-1)^k| + \frac{1}{k} \leq 1 + 1 = 2$. Hence the norm of the sequence $\mathbf{x}_k$ is bounded by $\sqrt{5}$; thus $\mathbf{x}_k$ is a bounded sequence. Therefore Bolzano-Weierstrass theorem implies that the sequence $\mathbf{x}_k$ contains a convergent subsequence. $\square$

6. [15 points] Suppose that $(\mathbf{x}_k)$ and $(\mathbf{y}_k)$ are sequences in $\mathbb{R}^n$ such that $(\mathbf{x}_k)$ converges to zero and $(\mathbf{y}_k)$ is bounded. Show that the limit of the sequence $(\mathbf{x}_k \cdot \mathbf{y}_k)$ is zero.

Solution. See Homework # 2.