MATHEMATICS 3220-1. Final Exam (Sample): solutions.

May 6, 2002

The exam is "closed book, closed notes". All problems should be treated as problems about "proofs"; just the correct computation without proper justification can result in a very low score on the problem.

1. [10 points] Using the definition of the limit of a sequence prove that the following sequence converges:

$$\mathbf{x}_k = (\frac{k}{k^2-1}, \frac{(-1)^k}{k+1}).$$

Solution. I claim that the limit of this sequence is $(0,0)$. To prove this, given $\epsilon > 0$ we have to find $n_0 \geq 2$ such that for all $k \geq n_0$ we have:

$$\frac{k}{k^2-1} < \epsilon, \frac{1}{k+1} < \epsilon.$$

Solving the second inequality we get:

$$\frac{1}{\epsilon} < k+1 \iff \frac{1}{\epsilon} - 1 < k.$$

Solving the first inequality we get:

$$\frac{k}{\epsilon} < k^2 - 1 \iff k^2 - \frac{k}{\epsilon} - 1 > 0,$$

the latter is staisfied for

$$k > \frac{1}{2}(\frac{1}{\epsilon} + \sqrt{\frac{1}{\epsilon^2} + 4}).$$

Let

$$M = \max(\frac{1}{\epsilon} - 1, \frac{1}{2}(\frac{1}{\epsilon} + \sqrt{\frac{1}{\epsilon^2} + 4})),$$

then $n_0 = [M] + 2$ will do the job. □

2. [15 points] Compute the following limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2}$$

(you can use limit theorems).

Solution

$$\frac{y^4}{x^2+y^2} = y^2 \frac{y^2}{x^2+y^2}.$$

Since $\frac{y^2}{x^2+y^2} \leq 1$, we get:

$$0 \leq \frac{y^4}{x^2+y^2} \leq y^2.$$

Since $\lim_{(x,y)\to(0,0)} y^2 = 0$, by squeeze lemma we get:

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = 0. \quad \square$$

3. [15 points] State the definition of a convex set and prove that the set

$$E = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

is convex.

Solution. See the textbook.

4. [10 points] State the implicit function theorem.

Solution. See the textbook.

5. [15 points] Prove the theorem on sequential compactness of closed and bounded subsets of $\mathbb{R}^n$. (Prove the theorem only the direction "If E is closed and bounded then...".)

Solution. See the textbook.

6. [10 points] Prove that the function

$$f(x,y) = \begin{cases} \frac{x}{y}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

is not differentiable at $(0,0)$.

Solution. I claim that f is not even continuous at zero: limit along the line $x = y$ gives us:

$$\lim_{x\to 0} \frac{x}{x} = 1 \neq f(0,0) = 0.$$

Since f is not continuous at zero, it is also non-differentiable at zero.

7. [15 points] Let $f(x,y) = (x, y^2)$. Using the definition of total derivative verify that

$$Df(x,y) = L = \begin{bmatrix} 1 & 0 \\ 0 & 2y \end{bmatrix}.$$

Solution. We have to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(x+h_1, y+h_2) - f(x,y) - L(h)}{\|\mathbf{h}\|} = 0.$$

Considering the numerator we get:

$$(x+h_1, (y+h_2)^2) - (x, y^2) - (h_1, 2yh_2) = (0, h_2^2).$$

So, we have to prove that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{h_2^2}{\sqrt{h_1^2+h_2^2}} = 0.$$

We have:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{h_2^2}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \sqrt{\frac{h_2^4}{h_1^2+h_2^2}} = 0,$$

by Problem # 2. $\square$

8. [10 points] Let $E = \{(x,y) : y \geq 0\}$. Determine if the subset $A \subset E$, $A = \{(x,0) : -2 \leq x \leq 2\}$, is relatively closed, relatively open or neither.

Solution. 1. I claim that A is relatively closed. Indeed, $A \subset \mathbb{R}^2$ is the intersection

$$A = \{(x, y) : y = 0\} \cap \{(x, y) : -2 \leq x \leq 2\}$$

The first set is given by an equation with continuous left hand side and the second set is given by 2 nonstrict inequalities with continuous function. Hence A is the intersection of two closed sets, hence it is closed. Since $A = A \cap E$, A is relatively closed in E as well.

2. I claim that A is not relatively open in E. Indeed, consider $\mathbf{x}_k = (0, \frac{1}{k}) \in E \setminus A$, $\lim_{k\to\infty} \mathbf{x}_k = (0, 0) \in A$. Thus A is not relatively open. □