MATHEMATICS 3220-1. Final Exam (Sample).

May 6, 2002

The exam is "closed book, closed notes". All problems should be treated as problems about "proofs"; just the correct computation without proper justification can result in a very low score on the problem.

1. [10 points] Using the definition of the limit of a sequence prove that the following sequence converges:

$$\mathbf{x}_k = (\frac{k}{k^2-1}, \frac{(-1)^k}{k+1}).$$

2. [15 points] Compute the following limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2}$$

(you can use limit theorems).

3. [15 points] State the definition of a convex set and prove that the set

$$E = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$$

is convex.

4. [10 points] State the implicit function theorem.

5. [15 points] Prove the theorem on sequential compactness of closed and bounded subsets of $\mathbb{R}^n$. (Prove the theorem only the direction "If E is closed and bounded then...".)

6. [10 points] Prove that the function

$$f(x,y) = \begin{cases} \frac{x}{y}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

is not differentiable at $(0,0)$.

7. [15 points] Let $f(x,y) = (x, y^2)$. Using the definition of total derivative verify that

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 2y \end{bmatrix}.$$

8. [10 points] Let $E = \{(x,y) : y \geq 0\}$. Determine if the subset $A \subset E$, $A = \{(x,0) : -2 \leq x \leq 2\}$, is relatively closed, relatively open or neither.