MATHEMATICS 3210-2. First Midterm Test (Sample).

January 25, 2002

The exam is "closed book, closed notes". All problems should be treated as problems about "proofs"; just the correct computation without proper justification can result in a very low score on the problem.

1. [15 points] Using the definition of the limit of a sequence prove that the following sequence converges:

$$\mathbf{x}_k = (\frac{k}{k-1}, \frac{1}{k^2}).$$

2. [15 points] State the Bolzano-Weierstrass theorem for $\mathbb{R}^n$.

3. [20 points] Compute the following limit or should that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2}$$

(you can use limit theorems).

4. [15 points] Let $f(x,y) = (x, y^2)$. Using the definition of total derivative verify that

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 2y \end{bmatrix}.$$

5. [15 points] Determine if the subset $\{(x,y) : x = 0, y \in \mathbb{R}\}$ of $\mathbb{R}^2$ is open. Give a proof!

6. [20 points] Reorder the following sentances to get a valid proof of the theorem on uniqueness of limit of a function:

Proof. By the triangle inequality we get:

$$\|\mathbf{v}-\mathbf{w}\| \leq \|f(\mathbf{x})-\mathbf{v}\| + \|f(\mathbf{x})-\mathbf{w}\| < 2\epsilon = \|\mathbf{v}-\mathbf{w}\|.$$

Suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{v}, \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{w}$. Suppose that $\mathbf{v} \neq \mathbf{w}$, then $\|\mathbf{v}-\mathbf{w}\| > 0$. Thus $\|\mathbf{v}-\mathbf{w}\| < \|\mathbf{v}-\mathbf{w}\|$. Contradiction. Hence for all $\mathbf{x}$ satisfying $\|\mathbf{x}-\mathbf{a}\| < \min(\delta_1, \delta_2)$ we have:

$$\|f(\mathbf{x})-\mathbf{v}\| < \epsilon, \|f(\mathbf{x})-\mathbf{w}\| < \epsilon.$$

Let $\epsilon = \|\mathbf{v} - \mathbf{w}\|/2$. Then (since $\lim_{x \to a} f(x) = \mathbf{v}$), there exists $\delta_1 > 0$ such that $0 < |\mathbf{x} - \mathbf{a}\| < \delta_1$ implies $\|f(\mathbf{x}) - \mathbf{v}\| < \delta_1$. Similarly, (since $\lim_{x \to a} f(x) = \mathbf{w}$), there exists $\delta_2 > 0$ such that $0 < |\mathbf{x} - \mathbf{a}\| < \delta_2$ implies $\|f(\mathbf{x}) - \mathbf{w}\| < \delta_2$. $\square$