MATHEMATICS 3220. Homework # 9: Solutions.

§11.2, # 4. [10 points]. Prove that

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{\sin\sqrt{x^2+y^2}}, & \text{if } 0 < \|(x,y)\| < \pi, \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is not differentiable at 0.

Solution. First, let's find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)}{x} = \lim_{x\to 0} \frac{x^2}{x\sin|x|} = \lim_{x\to 0} = \lim_{x\to 0} \frac{x}{\sin|x|}.$$

However

$$\lim_{x\to 0+} \frac{x}{\sin|x|} = \lim_{x\to 0} \frac{x}{\sin x} = 1,$$

$$\lim_{x\to 0-} \frac{x}{\sin|x|} = \lim_{x\to 0} \frac{x}{\sin(-x)} = -1,$$

(for instance, by l'Hospital rule or you can use the definition of derivative of $\sin(t)$ at zero). Hence

$$\lim_{x\to 0} \frac{x}{\sin|x|}$$

does not exist. Thus $f_x(0,0)$ does not exist, hence f is not differentiable at $(0,0)$. □

§11.3, # 2 (a). [15 points] Let $V \subset \mathbb{R}$ be open, $a \in V$ and $f, g : V \to \mathbb{R}^3$ are differentiable at a. Prove that $f \times g$ is differentiable at a and that

$$(f \times g)'(a) = f'(a) \times g(a) + f(a) \times g'(a).$$

Hint: try to use properties of the cross product (theorem 8.9 and remark 8.10) instead of the definition of the cross product.

Solution. Since f and g are differentiable at a, by lemma 11.21 there are functions $\epsilon, \delta : \mathbb{R} \to \mathbb{R}^3$ such that

$$\lim_{t\to 0} \epsilon(t) = \mathbf{0}, \lim_{t\to 0} \delta(t) = \mathbf{0}$$

and for all sufficiently small $t \in \mathbb{R}$ we have:

$$f(a+t) = f(a) + f'(a)t + |t|\epsilon(t), g(a+t) = g(a) + g'(a)t + |t|\delta(t).$$

Thus (see theorem 8.9)

$$f(a+t)\times g(a+t) - f(a)\times g(a) = (f(a)+f'(a)t+|t|\epsilon(t))\times(g(a)+g'(a)t+|t|\delta(t)) - f(a)\times g(a) =$$

$$= (f(a)\times g'(a) + f'(a)\times g(a))t + t^2 f'(a)\times g'(a) +$$

$$(f(a) + f'(a)t + |t|\epsilon(t)) \times |t|\delta(t) + |t|\epsilon(t) \times (g(a) + g'(a)t + |t|\delta(t)) =$$

$$(f(a)\times g'(a) + f'(a)\times g(a))t + |t|[|t|f'(a)\times g'(a) + f(a+t)\times\delta(t) + \epsilon(t)\times g(a+t)].$$

We let

$$\eta(t) := |t|f'(a) \times g'(a) + f(a+t) \times \delta(t) + \epsilon(t) \times g(a+t).$$

Thus

$$f(a+t) \times g(a+t) - f(a) \times g(a) = (f(a) \times g'(a) + f'(a) \times g(a))t + |t|\eta(t).$$

According to lemma 11.21 it remains to show that

$$\lim_{t\to 0} \eta(t) = \mathbf{0}.$$

The function $\eta(t)$ is the sum of $|t|f'(a) \times g'(a)$,

$$f(a+t) \times \delta(t)$$

and

$$\epsilon(t) \times g(a+t).$$

First, note that

$$\lim_{t\to 0} |t|f'(a) \times g'(a) = f'(a) \times g'(a) \lim_{t\to 0} |t| = 0.$$

I consider now the limit of the second summand, the limit of the third is similar. By Remark 8.9,

$$\|f(a+t) \times \delta(t)\| = \|f(a+t)\| \; \|\delta(t)\| \sin(\theta(t)),$$

where $0 \leq \theta(t) \leq \pi$ is the angle between $f(a+t)$ and $\delta(t)$. Thus

$$0 \leq \|f(a+t) \times \delta(t)\| \leq \|f(a+t)\| \; \|\delta(t)\|.$$

Since the function f is differentiable at a it is also continuous at a; hence

$$\lim_{t\to 0} \|f(a+t)\| \; \|\delta(t)\| = \|f(a)\| \cdot 0 = 0,$$

since $\lim_{t\to 0} \delta(t) = \mathbf{0}$. Thus by squeeze lemma,

$$\lim_{t\to 0} \|f(a+t) \times \delta(t)\| = 0$$

and hence

$$\lim_{t\to 0} f(a+t) \times \delta(t) = \mathbf{0}.$$

Similarly,

$$\lim_{t\to 0} \epsilon(t) \times g(a+t) = \mathbf{0}$$

and thus

$$\lim_{t\to 0} \eta(t) = \mathbf{0}.$$

5. [10 points] Let $f, g : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Prove that $u(x,y) = f(xy)$ satisfies

$$xu_x - yv_y = 0,$$

and that $v(x,y) = f(x-y) + g(x+y)$ satisfies the *wave equation*

$$v_{xx} - v_{yy} = 0.$$

Solution. To compute the partial derivatives we use the chain rule:

$$u_x = f'(x,y)\frac{\partial}{\partial x}(xy) = yf'(x,y),$$

$$u_y = f'(x, y)\frac{\partial}{\partial y}(xy) = xf'(x, y).$$

Hence

$$xu_x - yv_y = xyf'(x, y) - yxf'(x, y) = 0.$$

For the second part of the problem we apply the chain rule twice:

$$v_x = f'(x - y) + g'(x + y), v_y = -f'(x - y) + g'(x + y).$$

$$v_{xx} = f''(x - y) + g''(x + y), v_{yy} = f''(x - y) + g''(x + y).$$

Hence

$$v_{xx} - v_{yy} = f''(x - y) + g''(x + y) - (f''(x - y) + g''(x + y) = 0. \quad \square$$