MATHEMATICS 3220. Homework # 8: Solutions.

1. [15 points] Rearrange the following sentences to get a valid proof of the following

Theorem. If H is a compact set and $f: H \to \mathbb{R}^m$ is a continuous function then f is uniformly continuous.

Solution.

Proof. The definition of uniform continuity is: For each $\epsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}, \mathbf{x}' \in H$ we have

$$\|\mathbf{x} - \mathbf{x}'\| < \delta \Rightarrow \|f(\mathbf{x}) - f(\mathbf{x}')\| < \epsilon.$$

Suppose that uniform continuity fails. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ there are $\mathbf{x}, \mathbf{x}' \in H$ so that

$$\|\mathbf{x} - \mathbf{x}'\| < \delta \text{ and } \|f(\mathbf{x}) - f(\mathbf{x}')\| \geq \epsilon.$$

Let's choose δ of the form $1/n$. Then for each $n \in \mathbb{N}$ there are $\mathbf{x}_n, \mathbf{x}'_n \in H$ such that

$$\|\mathbf{x}_n - \mathbf{x}'_n\| < 1/n \text{ and } \|f(\mathbf{x}_n) - f(\mathbf{x}'_n)\| \geq \epsilon.$$

Since H is compact, there is a subsequence $(\mathbf{x}_{n_j})_{j\in\mathbb{N}}$ which converges to some $\mathbf{x} \in H$. By the triangle inequality we have:

$$0 \leq \|\mathbf{x} - \mathbf{x}'_n\| \leq \|\mathbf{x} - \mathbf{x}_n\| + \|\mathbf{x}_n - \mathbf{x}'_n\| < \|\mathbf{x} - \mathbf{x}_n\| + 1/n.$$

Since $\lim_{j\to\infty} \|\mathbf{x} - \mathbf{x}_{n_j}\| = 0$, by applying squeeze lemma we get:

$$\lim_{n\to\infty} \|\mathbf{x} - \mathbf{x}'_{n_j}\| = 0 \Rightarrow \lim_{n\to\infty} \mathbf{x}'_{n_j} = \mathbf{x}.$$

Since f is continuous at $\mathbf{x} \in H$ we get:

$$\lim_{j\to\infty} f(\mathbf{x}_{n_j}) = f(x) = \lim_{j\to\infty} f(\mathbf{x}'_{n_j}).$$

Hence

$$\lim_{j\to\infty} \|f(\mathbf{x}_{n_j}) - f(\mathbf{x}'_{n_j})\| = 0.$$

However, $\|f(\mathbf{x}_{n_j}) - f(\mathbf{x}'_{n_j})\| \geq \epsilon > 0$ for each j. Thus the sequence $\|f(\mathbf{x}_{n_j}) - f(\mathbf{x}'_{n_j})\|$ cannot converge to zero. Contradiction. Hence f is uniformly continuous on H. □

§11.2, # 3. [10 points] Prove that the function $\sqrt{|xy|}$ is not differentiable at $(0,0)$.

Solution. Suppose that f is differentiable. Then there is a linear map $L: \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{\mathbf{h}\to\emptyset} \frac{|f(\mathbf{h}) - L(\mathbf{h})|}{\|\mathbf{h}\|} = 0.$$

Consider this limit along the line $x = y$, i.e. $\mathbf{h} = (t,t)$, $t \in \mathbb{R} \setminus \{0\}$. Then we get:

$$\lim_{t\to 0} \frac{|\sqrt{t^2} - tL(1,1)|}{|t|} = 0.$$

Let $L(1,1) = a \in \mathbb{R}$. Note that $\sqrt{t^2} = |t|$, hence we get:

$$\lim_{t\to 0} |1 - \frac{t}{|t|}a| = 0,$$

equivalently,

$$\lim_{t \to 0} \frac{t}{|t|} a = 1.$$

Note that if $a = 0$ then the LHS is zero, which is impossible. Thus, $a \neq 0$. By taking limits from the right and from the left we get:

$$\lim_{t \to 0+} \frac{t}{|t|} a = \lim_{t \to 0+} (a) = a,$$

$$\lim_{t \to 0-} \frac{t}{|t|} a = \lim_{t \to 0+} (-a) = -a,$$

Since $a \neq -a$, the limit

$$\lim_{t \to 0} \frac{t}{|t|} a$$

does not exist. Thus the function f is not differentiable at $(0, 0)$.

Here is an alternative argument. First, let's compute $f_x(0, 0)$ and $f_y(0, 0)$:

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} 0 = 0,$$

$$f_y(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y \to 0} 0 = 0.$$

Assume now that f is differentiable at $(0, 0)$, then $Df(0, 0) = (0, 0)$, hence

$$\lim_{\mathbf{h} \to \emptyset} \frac{|f(\mathbf{h})|}{\|\mathbf{h}\|} = 0.$$

We would get the same limit by considering $\mathbf{h} = (t, t)$ where $t > 0$. However

$$\lim_{t \to \emptyset+} \frac{|f(t,t)|}{t} = \lim_{t \to \emptyset} \frac{t}{t} = 1.$$

Since $1 \neq 0$ we get a contradiction.

□