MATHEMATICS 3220. Homework # 7: Solutions.

§9.4, # 2(b). [10 points] Let $f(x) = \sqrt{x}$ and $g(x) = 1/x$ if $x \neq 0$ and $g(0) = 0$. Find $f^{-1}(E)$ and $g^{-1}(E)$ for $E = (-1, 1)$, $E = [-1, 1]$ and explain your results.

Solution. (1) Cosider first the function $f(x)$ defined on $H = [0, \infty)$. This function is strictly increasing and continuous, thus it admits a continuous inverse (which is the function $f^{-1}(y) = h(y) = y^2$, $y \geq 0$). Hence $f^{-1}(E) = h(E \cap [0, \infty))$. Since the function h is strictly increasing, continuous, the image of the interval is an interval. Moreover,

If E is a compact interval $[a, b]$, then $h(E)$ is also a compact interval $[f(a), f(b)]$. In the case $E = [-1, 1]$ we get: $f^{-1}(E) = f^{-1}([0, 1]) = h([0, 1]) = [0^2, 1^2] = [0, 1]$. Thus

$$f^{-1}((-1, 1)) = f^{-1}([0, 1)) = h([0, 1)) = h([0, 1]) \setminus \{h(1)\} = [0, 1).$$

(2) Secondly, consider the function $g(x)$. This function is not continuous, so we cannot directly apply theorems about continuous functions. However, I claim that the function

$$g : \mathbb{R} \to \mathbb{R}$$

has inverse which is g itself. Indeed, $g|_{\mathbb{R}\setminus\{0\}} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ has inverse which is $y \mapsto 1/y$. Also, $g(0) = 0 \notin \mathbb{R} \setminus \{0\}$. Hence for each $y \neq 0$, $g^{-1}(y) = 1/y$, for $y = 0$, $g^{-1}(0) = 0$. Thus $g = g^{-1}$. Therefore $g^{-1}(E) = g(E)$.

Consider $g((-1, 1)) = g((0, 1)) \cup g(0) \cup g((-1, 0))$. Since $(0, 1)$ is an interval and g is strictly decreasing and continuous on this interval, the image of $(0, 1)$ is an interval (call it I) as well. This interval I would have to be open since it is also $g^{-1}((0, 1))$, the interse image of an open interval under continuous function (namely, $g|_{(0,\infty)}$). Thus $I = (\inf_{x \in E} g(x), \sup_{x \in E} g(x)) = (\lim_{x \to 1-} g(x), \lim_{x \to 0+} g(x)) = (1, \infty)$. Similar arguments show that $g((-1, 0)) = (-\infty, -1)$. Thus

$$g((-1, 1)) = (-\infty, -1) \cup \{0\} \cup (1, \infty)$$

Therefore,

$$g([-1, 1]) = g((-1, 1)) \cup \{g(-1), g(1)\} = (-\infty, -1] \cup \{0\} \cup [1, \infty)$$

□

§9.4, # 6. [10 points] Let E be a connected subset of a Euclidean space. If $f : E \to \mathbb{R}$ is continuous, $\mathbf{a}, \mathbf{b} \in E$ are such that $f(\mathbf{a}) \neq f(\mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in E$ and y is a number which lies between $f(\mathbf{a})$ and $f(\mathbf{b})$, prove that there is $\mathbf{x} \in E$ such that $f(\mathbf{x}) = y$.

Proof. Suppose that $f(\mathbf{a}) < f(\mathbf{b})$ (the other case is done by interchanging $\mathbf{a}$ and $\mathbf{b}$). Since E is connected and f is continuous, the image $f(E)$ is also connected (theorem 9.36). Each connected subset of $\mathbb{R}$ is an interval (theorem 9.28). Hence $f(E) = I$ is an interval. Since $f(\mathbf{a}), f(\mathbf{b}) \in I$ and I is an interval, $f(\mathbf{a}) \leq y \leq f(\mathbf{b})$ implies that $y \in I$. Thus $y \in f(E)$, hence there is $\mathbf{x} \in E$ such that $f(\mathbf{x}) = y$. □

§9.4, # 7 (a). [10 points] Let $H \subset \mathbb{R}^n$ be a nonempty compact subset and $f : H \to \mathbb{R}^m$ be a continuous function. Prove that

$$\|f\|_H := \sup_{x \in H} \|f(x)\|$$

is finite and there exists $\mathbf{x}_0 \in H$ such that $\|f(\mathbf{x}_0)\| = \|f\|_H$.

Proof. Consider the function $g(\mathbf{y}) = \|\mathbf{y}\|$, $g : \mathbb{R}^m \to \mathbb{R}$. Then the function g is continuous. Thus $h(\mathbf{x}) := \|f(\mathbf{x})\| = g \circ f(\mathbf{x})$ is the composition of the continuous functions f and g. Thus the function $h : H \to \mathbb{R}$ is continuous. Hence, by theorem 9.38, this function attains its maximal value:

There exists $\mathbf{x}_0 \in H$ such that $-\infty < h(\mathbf{x}_0) = \sup_{x \in H} h(x) < \infty$. Thus $h(\mathbf{x}_0) = \|f\|_H$. □

§11.1, # 3 (a). [10 points] For the following function f compute f_x and determine where it is continuous.

$$f(x,y) = \begin{cases} \frac{x^4+y^4}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Solution. At each $(x,y) \neq (0,0)$ we have:

$$f_x(x,y) = \frac{4x^3(x^2+y^2) - (x^4+y^4)(2x)}{(x^2+y^2)^2} = \frac{2x^5 + 4x^3y^2 - 2xy^4}{(x^2+y^2)^2}.$$

At $(0,0)$ we get the partial derivative:

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - 0}{x} = \lim_{x \to 0} \frac{x^4}{x^3} = 0.$$

The function $\frac{2x^5+4x^3y^2-2xy^4}{(x^2+y^2)^2}$ is a rational function with the domain $\mathbb{R}^2 \setminus \{(0,0)\}$, hence it is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus, it remains to verify if the function f_x is continuous at the origin. Namely, we would like to compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{2x^5 + 4x^3y^2 - 2xy^4}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} x\frac{2x^4 + 4x^2y^2 - 2y^4}{(x^2+y^2)^2}$$

I claim that the fraction

$$R = \frac{2x^4 - 4x^2y^2 - 2y^4}{(x^2+y^2)^2} = \frac{2x^4}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2} - \frac{2y^4}{(x^2+y^2)^2}$$

is bounded. Consider first

$$A = \frac{2x^4}{(x^2+y^2)^2} = 2\left(\frac{x^2}{x^2+y^2}\right)^2.$$

Since $x^2 \leq x^2 + y^2$, $\frac{x^2}{x^2+y^2} \leq 1$. Hence $A \leq 2$. Next,

$$B = \frac{4x^2y^2}{(x^2+y^2)^2} = 4\frac{x^2}{x^2+y^2}\frac{y^2}{x^2+y^2} \leq 4$$

for the same reason as above. Finally,

$$C = \frac{2y^4}{(x^2+y^2)^2} = 2\left(\frac{y^2}{x^2+y^2}\right)^2 \leq 2.$$

Thus $|R| \leq A + B + C \leq 2 + 4 + 2 = 8$. Thus

$$|R| = \left|\frac{2x^4 + 4x^2y^2 - 2y^4}{(x^2+y^2)^2}\right| \leq 8.$$

Thus in the limit

$$\lim_{(x,y)\to(0,0)} x\frac{2x^4+4x^2y^2-2y^4}{(x^2+y^2)^2}$$

we have the product of the function x (which converges to zero) and the bounded function $R = \frac{2x^4-4x^2y^2-2y^4}{(x^2+y^2)^2}$. Thus the limit equals zero. Hence

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0).$$

Thus the function f_x is continuous at zero. Hence f_x is continuous everywhere in $\mathbb{R}^2$. $\square$

§11.1, # 5 (a). [5 points] Evaluate the following expression:

$$\lim_{y\to 0}\int_0^1 \cos(x^2y+xy^2)dx.$$

Solution. The function $f(x,y) = \cos(x^2y+xy^2)$ is continuous on $\mathbb{R}^2$ (as composition of two continuous functions). Hence we can apply theorem 11.4 and we get:

$$\lim_{y\to 0}\int_0^1 \cos(x^2y+xy^2)dx = \int_0^1 f(x,0)dx = \int_0^1 \cos(0)dx = 1. \quad \square$$