MATHEMATICS 3220. Homework # 6: Solutions.

1. §9.2, # 1. Identify which of the following sets is compact and which is not. If E is not compact, find the smallest compact subset H (if there is one) such that $E \subset H$.

(a) [5 points] $E = \{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\}$.

Solution. E is bounded since for each $x \in E$, $0 \le x \le 1$. To see that E is closed note that

$$E^c = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{k=1}^{\infty} (\frac{1}{k+1}, \frac{1}{k}).$$

This is a union of open sets, hence it is open. Thus E is closed. Hence E is compact.

(b) [5 points] $E = \{(x, y) \in \mathbb{R}^2 : a^2 \le x^2 + y^2 \le b^2\}$ for real numbers $0 < a < b$.

Solution. E is bounded, since for each $\mathbf{x} \in E$, $\|\mathbf{x}\| \le b$. On the other hand, this set is closed since it is the intersection of two closed sets $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \ge a^2\}$ and $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le b^2\}$. Each of these sets is given by a nonstrict inequality with continuous left hand side. Thus both sets are closed, hence E is closed as well. Thus E is both closed and bounded, hence it is compact. □

(d) [5 points] $E = \{(x, y) \in \mathbb{R}^2 : |xy| \le 1\}$.

Solution. This set is unbounded, since it contains the x-axis. Hence, each set H containing E is unbounded as well. Thus, there is no compact set H which would contain E. □

2. [10 points] Let A, B be compact subsets of $\mathbb{R}^n$. Prove that $A \cup B$ and $A \cap B$ are compact.

Solution. Recall that "compact" in $\mathbb{R}^n$ is the same as "closed and bounded". Let's verify that $A \cup B$ and $A \cap B$ are both closed and bounded. Recall that the union of a finite number of closed sets is again closed; hence $A \cup B$ is closed. Since the intersection of closed sets is closed, we conclude that $A \cap B$ is closed.

Since A and B are bounded, there are numbers $M, N > 0$ such that for each $\mathbf{x} \in A$, $\|\mathbf{x}\| \le M$ and for each $\mathbf{y} \in B$, $\|\mathbf{y}\| \le N$. Let $L = \max(M, N)$. Then for each $\mathbf{x} \in A \cup B$, $\|\mathbf{x}\| \le L$. Thus $A \cup B$ is bounded. Since $A \cap B \subset A \cup B$, the set $A \cap B$ is bounded as well.

Thus both sets $A \cup B$ and $A \cap B$ are both closed and bounded; hence they are both compact. □

Here is another proof, based upon the definition of sequential compactness. Let $\mathbf{x}_k \in A \cup B$ be a sequence. Let K denote the set of those indices for which $\mathbf{x}_k \in A$ and let L denote the set of those indices for which $\mathbf{x}_k \in B$. Note that $K \cup L = \mathbb{N}$. Thus either K or L is infinite. Suppose that K is infinite. Then we get a subsequence

$$(\mathbf{x}_k)_{k \in K}$$

of the original sequence $\mathbf{x}_k$ such that each member of this subsequence is in A. Since A is compact, we can find a subsequence in $(\mathbf{x}_k)_{k \in K}$ which converges in A. Thus we have constructed a subsequence in $(\mathbf{x}_k)$ which has a limit in $A \cup B$.

To prove that $A \cap B$ is compact note that $A \cap B$ is closed and is a subset of the compact set A. Hence $A \cap B$ is compact (remark 9.21). □

3. [5 points] Suppose that $E \subset \mathbb{R}$ is compact and nonempty. Prove that $\sup(E) \in E, \inf(E) \in E$.

Solution. First of all, since E is compact, it is closed and bounded. Since E is a nonempty, bounded subset of $\mathbb{R}$, $M = \sup(E) \in \mathbb{R}$ and $m = \inf(E) \in \mathbb{R}$. Note that M is the least upper bound for E. If $M \notin E$, then, since E^c is open, there is an open interval $(M - \epsilon, M + \epsilon)$ around M which is contained in E^c. Hence $M - \epsilon/2$ is also an upper bound for E, which contradicts the assumption that $M = \sup(E)$. Thus $M \in E$. The same argument works for m. □

2, §9.3. (a) [10 points] Sketch a graph of the set $A = \{(x, y) : x^2 + 2y^2 < 6, y \geq 0\}$ and decide if this set is relatively open or relatively closed in $E = \{(x, y) : y \geq 0\}$. Do the same for the set $G = \{(x, y) : x^2 + 2y^2 < 6\}$.

Solution. 1. Consider the set E. Then A equals

$$E \cap \{(x, y) : x^2 + 2y^2 < 6\}.$$

The set $U = \{(x, y) : x^2 + 2y^2 < 6\}$ is open (since given by a strict inequality with continuous left hand side). Thus $E \cap U = A$ is relatively open in E.

2. Consider the set A as the subset of G. Then A equals

$$G \cap \{(x, y) : y \geq 0\}.$$

The set $C = \{(x, y) : y \geq 0\}$ is closed, hence the intersection $A = G \cap C$ is relatively closed in G. □

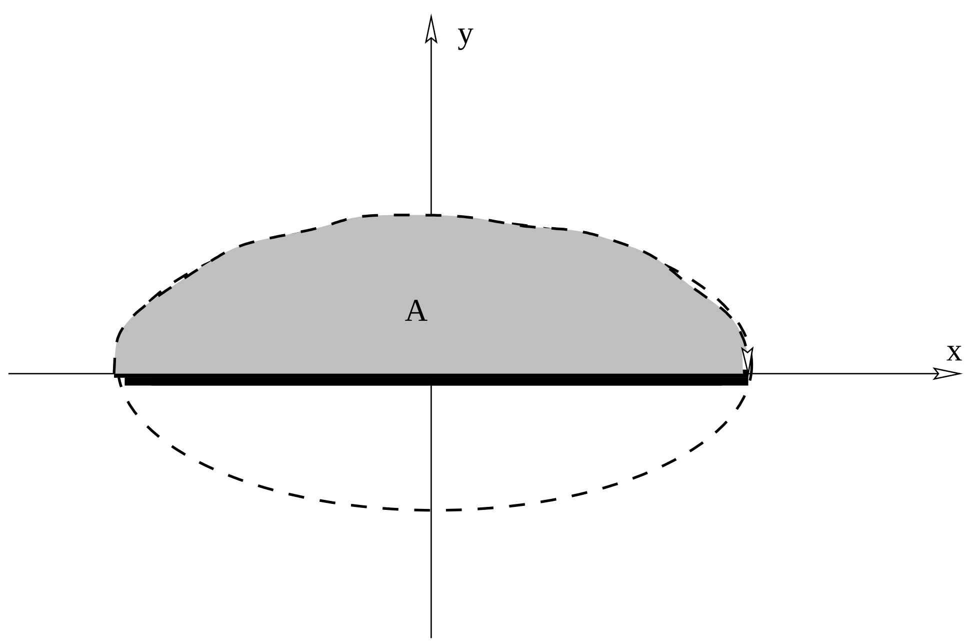

Figure 1: Problem 2 (a), §9.3.

(b) [10 points] Sketch a graph of the set $A = \{(x, y) : x^2 + y^2 \leq 1, (x-2)^2 + y^2 < 2\}$ and decide if this set is relatively open or relatively closed in the set $E = \overline{B_1(0, 0)}$. Do the same for A as a the subset of $G = B_{\sqrt{2}}(2, 0)$.

Solution.

(1) I claim that A is relatively open in E. To see this note that

$$A = E \cap \{(x, y) : (x - 2)^2 + y^2 < 2\}.$$

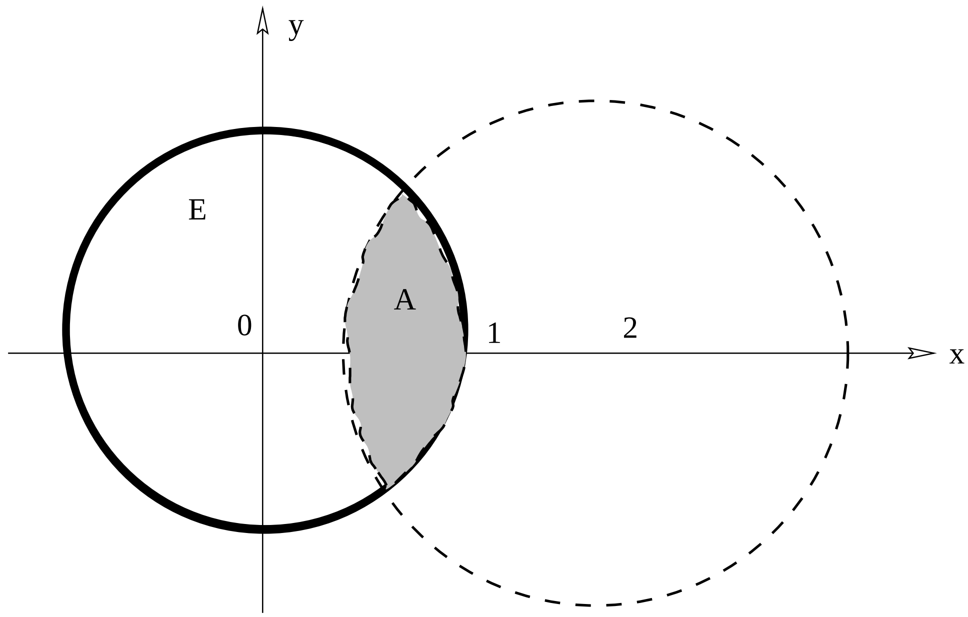

Figure 2: Problem 2 (b), §9.3.

The set $U = \{(x, y) : (x-2)^2 + y^2 < 2\}$ is open, hence A is relatively open in E.

(2) I claim that A is relatively closed in G. To see that A is relatively closed note that the inequality $(x-2)^2 + y^2 < 2$ describes the open ball of radius $\sqrt{2}$ centered at $(2, 0)$, which is $G = B_{\sqrt{2}}(2, 0)$. Hence $A = \{(x, y) : x^2 + y^2 \leq 1\} \cap G$. Since the set $V = \{(x, y) : x^2 + y^2 \leq 1\}$ is closed, the intersection $A = V \cap G$ is relatively closed in G.

To see that A is not relatively open consider the point $q = (1, 0) \in A$ and take a sequence of points $q_k = (1 + \frac{1}{k}, 0)$. Then for all $k \geq 3$ we have

$$1 < (1 + \frac{1}{k})^2 < 2,$$

hence $q_k \in G \setminus A$. The limit of the sequence q_k is $q \in A$. Thus $G \setminus A$ is not relatively closed, so A is not relatively open in G. □

3. [10 points] Prove that the intersection of connected subsets of $\mathbb{R}$ is connected. Show that this is false if $\mathbb{R}$ is replaced by $\mathbb{R}^2$.

Proof. (1) Recall that connected subsets of $\mathbb{R}$ are intervals (theorem 9.28). So, suppose that $A, B \subset \mathbb{R}$ are intervals. There are several cases to consider depending on the shape of the intervals, I will consider the case where both A and B are open intervals:

$$A = (a, b), B = (c, d).$$

Up to interchanging the names of the intervals, we can assume that $a \leq c$. Then

$$I = A \cap B = (c, \min(b, d)),$$

which is again an interval (possibly empty, if $b \leq c$). Since I is an interval, it is connected, by theorem 9.28. The argument in the case of closed and half-closed intervals is the same.

(2) Let's show that the assertion is false if we replace $\mathbb{R}$ by $\mathbb{R}^2$. Consider for instance, the circle $C = \{(x,y) : x^2+y^2 = 1\}$ and the line L which is the x-axis. Then line L is connected by theorem 9.28. The circle C is the image of the continuous map $f(t) = (\sin(t), \cos(t))$, $t \in \mathbb{R}$. Since $\mathbb{R}$ is connected, the circle C is connected as well. On the other hand, the intersection $A = C \cap L = \{(-1,0), (1,0)\} = \{p, q\}$. To see that A is disconnected consider the open disks $B_1(p), V = B_1(q)$. Then $B_1(p) \cap A = \{p\}$, $B_1(q) \cap A = \{q\}$, hence the pair of sets $\{\{p\}, \{q\}\}$ forms a separation of A; hence A is not connected. □

6. [10 points] Suppose that $\{E_\alpha : \alpha \in A\}$ is a collection of connected subsets of $\mathbb{R}^n$, whose intersection is nonempty. Show that $E = \cup_{\alpha \in A} E_\alpha$ is connected.

Proof. Let $x \in \cap_{\alpha \in A} E_\alpha$. Suppose that E is not connected, so there is a separation $\{U, V\}$ of E. Since $E = U \cup V$, either $x \in U$ or $x \in V$. Suppose that $x \in U$. Then $x \in U \cap E_\alpha$ for each $\alpha \in A$, hence $U \cap E_\alpha$ is nonempty. Note that both sets $U_\alpha := U \cap E_\alpha$ and $V_\alpha := V \cap E_\alpha$ are relatively open in E_α. Also, since $U \cap V = \emptyset$, $U_\alpha \cap V_\alpha = \emptyset$; since $U \cup V = E$, $U_\alpha \cup V_\alpha = E_\alpha$. Thus $\{U_\alpha, V_\alpha\}$ form a separation of E_α, unless $V_\alpha = \emptyset$. Since E_α is connected, it does not admit a separation. Thus $V_\alpha = \emptyset$ for each $\alpha \in A$. However $V = \cup_{\alpha \in A} V_\alpha$, so we proved that $V = \emptyset$. This however contradicts the assumption that $\{U, V\}$ is a separation of E. This proves that E is connected. □