MATHEMATICS 3220. Homework # 5: Solutions.

In all the problems the students are allowed to use the following:

Theorem. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is a continuous function and* $c \in \mathbb{R}$. *Then the sets* $\{\mathbf{x} : f(\mathbf{x}) > c\}, \{\mathbf{x} : f(\mathbf{x}) < c\}$ *are open and the sets* $\{\mathbf{x} : f(\mathbf{x}) \geq c\}, \{\mathbf{x} : f(\mathbf{x}) \leq c\}$ *are closed.*

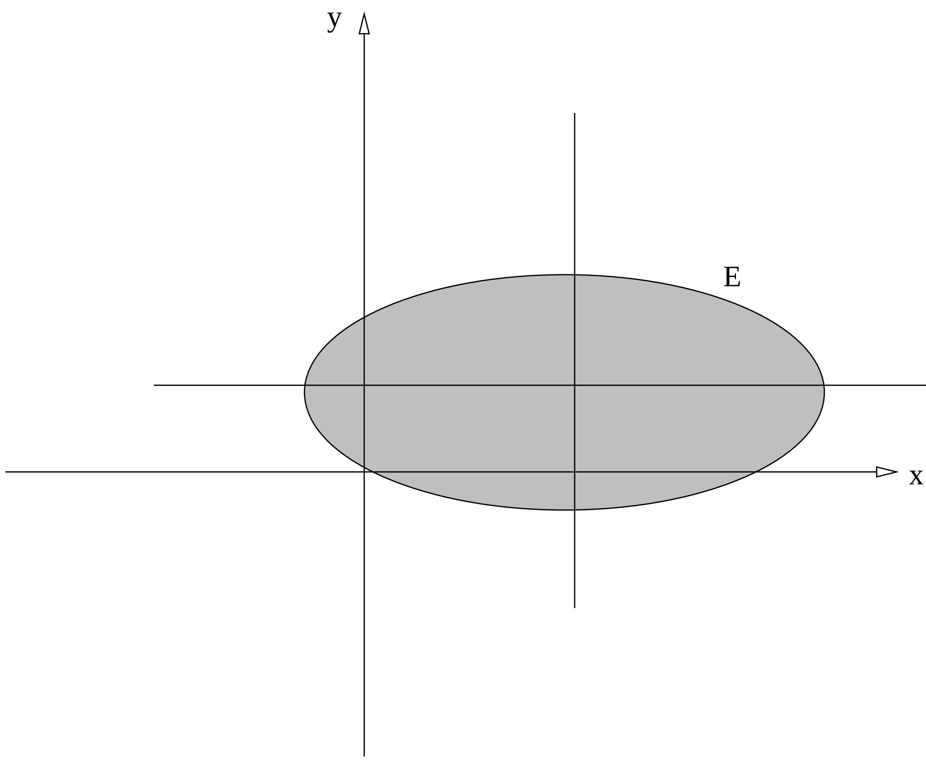
1. Find interior, closure and the boundary of the following sets. Sketch E. Give a proof that what you found are indeed interior, closure and the boundary.

a) [10 points] $E = \{(x, y) : x^2 + 4y^2 - 2x - 2y \leq 0\}$.

Solution. Consider the equation $x^2 + 4y^2 - 2x - 2y = 0$, it is equivalent to

$$(x - 1)^2 + 4(y - 0.25)^2 = 1.25.$$

The latter is the equation of an ellipse with the principal axes $x = 1$ and $y = 1/4$.



We first find the interior E^0 of E. I claim that each point $\mathbf{x}$ of the ellipse

$$A = \{(x, y) : (x - 1)^2 + 4(y - 0.25)^2 = 1.25\}$$

is a "bad point", i.e. is a limit point of the complement, i.e. there exists a sequence of points $\mathbf{x}_k \in E^c$ which converges to $\mathbf{x}$.

Case 1: $\mathbf{x} = (x, y)$, $x \geq 1$. Then take the sequence $\mathbf{x}_k = (x + \frac{1}{k}, y)$. It is clear that $\lim_{k\to\infty} \mathbf{x}_k = (x, y)$. To see that $\mathbf{x}_k \notin E$ note that

$$(x_k - 1)^2 + 4(y - .25)^2 = (x - 1 + \frac{1}{k})^2 + 4(y - .25)^2 =$$

$$= (x - 1)^2 + 4(y - .25)^2 + 2(x - 1)/k + \frac{1}{k^2} \geq 1.25 + \frac{1}{k^2} > 1.25.$$

Thus $\mathbf{x}_k \notin E$.

Case 2. $\mathbf{x} = (x, y)$, $x \le 1$. The proof is the same, just take $\mathbf{x}_k = (x - \frac{1}{k}, y)$.

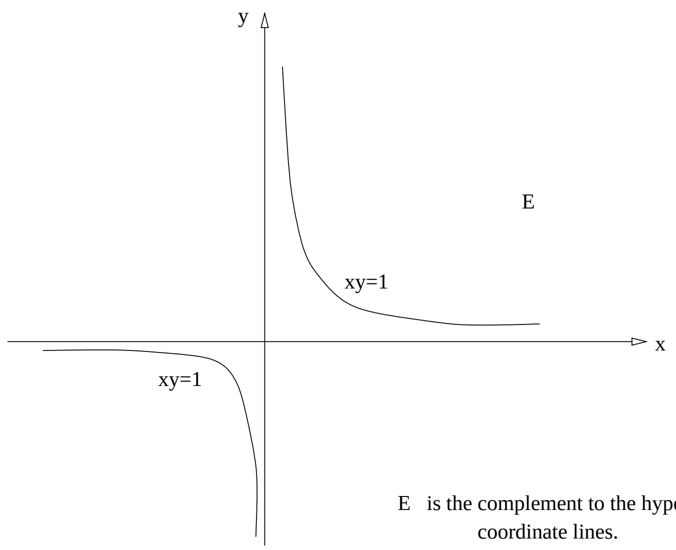
Thus we proved that each point $\mathbf{x}$ of the ellipse A is a limit point of the complement of E, i.e. $\mathbf{x} \notin E^0$. It remains to show that $E \setminus A$ is an open set. This set is $E \setminus A = \{(x, y) : x^2 + 4y^2 - 2x - 2y < 0\}$, given by strict inequality with continuous left hand side, thus $E \setminus A$ is open. Hence $\{(x, y) : x^2 + 4y^2 - 2x - 2y < 0\}$ is the largest open subset of E, hence the interior of E is

$$E^0 = \{(x, y) : x^2 + 4y^2 - 2x - 2y < 0\}.$$

To find the closure of E note that E is given by nonstrict inequality with continuous left hand side, thus E is closed. Hence $\bar{E} = E$. Finally, the boundary ∂E of E is

$$\bar{E} \setminus E^0 = \{(x, y) : x^2 + 4y^2 - 2x - 2y = 0\}. \quad \square$$

b) [10 points] $E = \{(x, y) : xy \neq 0, xy \neq 1\}$.



E is the complement to the hyperbola xy=1 and the union of two coordinate lines.

Solution. First of all, let's show that the set E is open. The set E is equals

$$(\{(x, y) : xy > 0\} \cup \{(x, y) : xy < 0\}) \cap (\{(x, y) : xy > 1\} \cup \{(x, y) : xy < 1\}).$$

Each of the sets $\{(x, y) : xy > 0\}$, $\{(x, y) : xy < 0\}$, $\{(x, y) : xy > 1\}$, $\{(x, y) : xy < 1\}$ is given by strict inequality with continuous left hand side, hence all these sets are open. Since the union of open sets is again open, the sets $(\{(x, y) : xy > 0\} \cup \{(x, y) : xy < 0\})$ and $(\{(x, y) : xy > 1\} \cup \{(x, y) : xy < 1\})$ are both open. Since the intersection of two open sets is open, the set E is open. Thus $E^0 = E$.

We now find the closure of E. I claim that each point of $\mathbb{R}^2$ belongs to the closure of E. It suffices to show that each point $\mathbf{x}$ in the set $E^c = \{(x, y) : xy = 0\} \cup \{(x, y) : xy = 1\}$ is the limit of a sequence $\mathbf{x}_k \in E$.

First, let us take a point $\mathbf{x} = (x, y)$ such that $xy = 0$. Take $\mathbf{x}_k = (x_k, y_k) = (x + \frac{1}{k}, y + \frac{1}{k})$. Then

$$\lim_{k \to \infty} (x_k, y_k) = (x, y),$$

hence

$$\lim_{k \to \infty} x_k y_k = 0.$$

In particular, for all sufficiently large values of k, $x_k y_k \neq 1$. It remains to show that $\mathbf{x}_k$ satisfies $x_k y_k \neq 0$ for all sufficiently large values of k. Indeed,

$$x_k y_k = (x + \frac{1}{k})(y + \frac{1}{k}) = \frac{x}{k} + \frac{y}{k} + \frac{1}{k^2} = \frac{1}{k}(x + y + \frac{1}{k}).$$

Note that for each (x, y) there is at most one $k = k_0$ such that $x + y + \frac{1}{k} = 0$. For all other k's we have $x + y + \frac{1}{k} \neq 0$. Since $\frac{1}{k} \neq 0$, we conclude that $\mathbf{x}_k \notin \{(x, y) : xy = 0\}$ for all $k \neq k_0$. Thus $\mathbf{x}$ is a limit point of E and so it belongs to the closure of E.

Secondly, take a point $\mathbf{x} = (x, y)$ such that $xy = 1$. Take $\mathbf{x}_k = (x_k, y_k) = (x + \frac{1}{k}, y + \frac{1}{k})$. Then

$$\lim_{k\to\infty} (x_k, y_k) = (x, y),$$

hence

$$\lim_{k\to\infty} x_k y_k = 1.$$

In particular, for all sufficiently large values of k, $x_k y_k \neq 0$. It remains to show that $\mathbf{x}_k$ satisfies $x_k y_k \neq 1$ for all sufficiently large values of k. Indeed,

$$x_k y_k - 1 = (x + \frac{1}{k})(y + \frac{1}{k}) - 1 = \frac{x}{k} + \frac{y}{k} + \frac{1}{k^2} = \frac{1}{k}(x + y + \frac{1}{k}).$$

But we already proved that the latter is nonzero for all k's with possibly one exception k_0. Hence $x_k y_k \neq 1$ for all $k \neq k_0$. Thus we conclude that $\mathbf{x}_k \notin \{(x, y) : xy = 1\}$ for all $k \neq k_0$. Hence $\mathbf{x}$ is a limit point of E and so it belongs to the closure of E. Therefore $\bar{E} = \mathbb{R}^2$. The boundary of E is

$$\partial E = \bar{E} \setminus E^0 = \{(x, y) : xy = 0, \text{ or } xy = 1\}. \quad \square$$

2. [5 points] Prove that the following set is open:

$$E = \{(x, y, z) : x^3 + y^7 - xzy + \sin(x + y + z) < 0, e^{xy} < 7\}.$$

Solution. Note that the function $x^3 + y^7 - xzy$ is a polynomial, hence it is continuous; the function $\sin(x + y + z)$ is continuous as a composition of two continuous functions. Thus the function $f(x, y, z) = x^3 + y^7 - xzy + \sin(x + y + z)$ is continuous. The function $g(x, y, z) = e^{xy}$ is a composition of two continuous functions, hence it is continuous too. The set E is

$$\{(x, y, z) : f(x, y, z) < 0\} \cap \{(x, y, z) : g(x, y, z) < 7\}.$$

Both sets $\{(x, y, z) : f(x, y, z) < 0\}$ and $\{(x, y, z) : g(x, y, z) < 7\}$ are given by strict inequalities with continuous left hand side, hence they are both open. Since the intersection of two open sets is again open, the set E is open as well. $\square$