MATHEMATICS 3220. Homework # 4.

1. §8.4, # 2, 6, 8.

2 [20 points] Let $f(x,y) = (xy, x+y, x^2-y^2)$. Using definition 8.31 prove that f is differentiable on $\mathbb{R}^2$ and its total derivative is given by

$$Df(x,y) = \begin{bmatrix} y & x \\ 1 & 1 \\ 2x & -2y \end{bmatrix}.$$

Let $(a,b) \in \mathbb{R}^2$. Let $\mathbf{h} = (x,y)$. Then to show that

$$Df(a,b) = T = \begin{bmatrix} b & a \\ 1 & 1 \\ 2a & -2b \end{bmatrix}$$

we have to verify that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

First, consider the vector in the numerator:

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h}) =$$

$$= \begin{bmatrix} (a+x)(b+y) \\ (a+x)+(b+y) \\ (a+x)^2-(b+y)^2 \end{bmatrix} - \begin{bmatrix} ab \\ a+b \\ a^2-b^2 \end{bmatrix} - \begin{bmatrix} bx+ay \\ x+y \\ 2ax-2by \end{bmatrix} =$$

$$\begin{bmatrix} xy \\ 0 \\ x^2-y^2 \end{bmatrix}.$$

The vector in the denominator is $\mathbf{h} = (x,y)$. We consider the ratio of the squares of the norms of the vectors $(xy, 0, x^2-y^2)$ and (x,y):

$$\frac{x^2y^2 + (x^2-y^2)^2}{x^2+y^2} = \frac{x^4+y^4-x^2y^2}{x^2+y^2} =$$

$$= \frac{(x^2+y^2)^2 - 3x^2y^2}{x^2+y^2} = \frac{(x^2+y^2-\sqrt{3}xy)(x^2+y^2+\sqrt{3}xy)}{x^2+y^2}.$$

Note that for $xy \geq 0$, $x^2+y^2-\sqrt{3}xy \leq x^2+y^2$; for $xy \leq 0$, $x^2+y^2+\sqrt{3}xy \leq x^2+y^2$. In the first case we get:

$$\frac{(x^2+y^2-\sqrt{3}xy)(x^2+y^2+\sqrt{3}xy)}{x^2+y^2} \leq x^2+y^2+\sqrt{3}xy.$$

In the second case we get:

$$\frac{(x^2+y^2-\sqrt{3}xy)(x^2+y^2+\sqrt{3}xy)}{x^2+y^2} \leq x^2+y^2-\sqrt{3}xy.$$

Hence

$$0 \le \frac{(x^2+y^2-\sqrt{3}xy)(x^2+y^2+\sqrt{3}xy)}{x^2+y^2} \le$$

$$\le \max(|x^2+y^2+\sqrt{3}xy|, |x^2+y^2-\sqrt{3}xy|).$$

Note that both $x^2+y^2+\sqrt{3}xy$ and $x^2+y^2-\sqrt{3}xy$ are polynomial functions, hence their limits as $(x,y) \to (0,0)$ are equal to zero. Hence by the squeeze theorem,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2-\sqrt{3}xy)(x^2+y^2+\sqrt{3}xy)}{x^2+y^2} = 0. \quad \square$$

6. Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$. (a) [10 points] Suppose that f and g are differentiable at a, prove that $f+g$ is also differentiable at a and

$$D(f+g)(a) = Df(a) + Dg(a).$$

Proof. Consider the limit:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) + g(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - g(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h}) - Dg(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|}.$$

We have to show that this limit is zero. To prove this note that by the triangle inequality,

$$\|f(\mathbf{a}+\mathbf{h}) + g(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - g(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h}) - Dg(\mathbf{a})(\mathbf{h})\| \le$$

$$\le \|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h})\| + \|g(\mathbf{a}+\mathbf{h}) - g(\mathbf{a}) - Dg(\mathbf{a})(\mathbf{h})\|.$$

Since

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|} = 0,$$

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|g(\mathbf{a}+\mathbf{h}) - g(\mathbf{a}) - Dg(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|} = 0,$$

by applying the theorem about the limit of the fum and the squeeze theorem we conclude that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) + g(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - g(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h}) - Dg(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|}.$$

$\square$

(b) Suppose that f is differentiable at a and $\alpha \in \mathbb{R}$. Prove that αf is differentiable at a and $D(\alpha f)(a) = \alpha Df(a)$. [10 points]

Proof. Consider the limit:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\alpha f(\mathbf{a}+\mathbf{h}) - \alpha f(\mathbf{a}) - \alpha Df(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|} =$$

$$\lim_{\mathbf{h}\to\mathbf{0}} |\alpha| \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|} =$$

$$|\alpha| \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|} = |\alpha| \cdot 0 = 0.$$

Hence $alDf(\mathbf{a})$ is indeed the derivative of αf at a. □

8. [10 points] Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function. Show that T is differentiable at each $\mathbf{a} \in \mathbb{R}^n$ and $DT(\mathbf{a}) = T$.

Proof. We have to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{a}+\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

To prove this note that $T(\mathbf{a}+\mathbf{h}) = T(\mathbf{a}) - T(\mathbf{h})$ (because T is linear), hence the numerator $\|T(\mathbf{a}+\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})\|$ is zero, therefore the limit is zero as well. □

2. §9.1, # 1 (a,b), # 2 (a,b). (We did not discuss the boundary yet, so do not find it.) I do not require any proofs for these problems (yet!). Just make a correct sketch (if required) and state what are the closure, interior, etc.

1 (a) [10 points]. Find the interior and closure of $[a, b)$, where $a < b$.

Proof. 1. The interior is the largest open subset of $[a, b)$. Note that (a, b) is an open set (this is a special case of theorem that open balls are open sets), so the interior should contain (a, b). On the other hand, $[a, b)$ is not open since for each $\epsilon > 0$ the ball $B_\epsilon(a)$ contains a point $a - \epsilon/2$ which is not in $[a, b)$. Hence the interior of $[a, b)$ is (a, b).

2. The closure is the smallest closed set which contains $[a, b)$. Since $[a, b]$ is closed (since this is a special case of a closed ball), the closure should be contained in $[a, b]$. On the other hand, the set $[a, b)$ is not closed (since its complement $(-\infty, a) \cup [b, \infty)$ is not open), hence the closure equals $[a, b]$. □

(b) $E = \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$. 1. The interior of this set is empty, since each ball $B_\epsilon(1/n)$ contains an irrational point, which therefore does not belong to E.

2. Note that $cl(E)$ contains E, it also contains the point 0, since 0 is the limit of a sequence of points $1/n$, each of which belongs to E. Note that the set

$$\mathbb{R}\backslash(E \cup \{0\}) = (-\infty, 0) \cup \cup_{n\in\mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, \infty).$$

Hence $\mathbb{R}\backslash(E \cup \{0\})$ is the union of open sets, hence it is open. Thus $E \cup \{0\}$ is closed. Since the closure is the smallest closed set containing E (it also must contain 0), we conclude that the closure equals $E \cup \{0\}$. □

2. Identify which of these sets are open, which are closed, which are neither. Find the interior and the closure and sketch E.

(a) [10 points] $E = \{(x, y) : x^2 + 4y^2 \leq 1\}$.

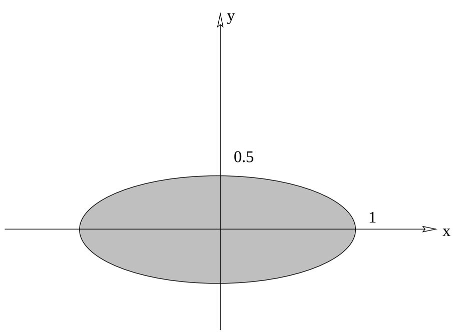
Solution. This set is closed and is not open. The closure of this set is E itself, the interior of E is the set

$$\{(x, y) : x^2 + 4y^2 < 1\}.$$

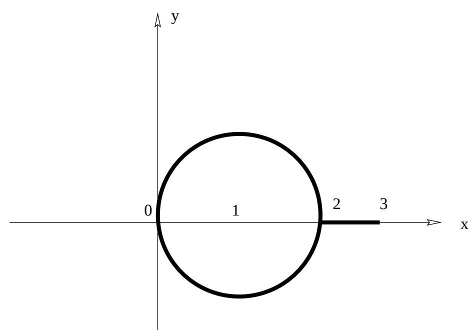
(b) [10 points] $E = \{(x, y) : x^2 - 2x + y^2 = 0\} \cup \{(x, 0) : x \in [2, 3]\}$.

Note that $x^2 - 2x + y^2 = 0 \iff (x - 1)^2 + y^2 = 1$. The latter is equation of the circle with center at $(1, 0)$ and radius 1. The set $\{(x, 0) : x \in [2, 3]\}$ is the straight line segment between $(2, 0)$ and $(3, 0)$.

The set E is closed and not open. Its interior is empty, its closure is the set E itself.



Part a.



Part b.