

MATHEMATICS 3220. Homework # 3.

1. [10 points] §8.3, # 1 (a). Find the domain of the function f , find the limit of f as $(x, y) \rightarrow (1, -1)$;

$$f(x, y) = \left(\frac{x-1}{y-1}, \frac{x^2+x-2}{x-1} \right).$$

Solution. The first component of the function f has the domain $\{(x, y) : y \neq 1\}$; the second component has the domain $\{(x, y) : x \neq 1\}$. Hence the domain of f is

$$\{(x, y) : x \neq 1, y \neq 1\}$$

(geometrically this is the 2-plane with two lines removed). Note that the point $P = (1, -1)$ belongs to one of these lines (namely, the horizontal line $x = 1$), hence there is no open ball $B_R(P)$ centered at P such that $B_R(P) \setminus \{P\} \subset \text{Dom}(f)$. So, technically speaking, if we accept Definition 8.19 literally, then the limit does not exist.

However, as we discussed in the class, we will be using only the assumption that there is a sequence $\mathbf{x}_k \in \text{Dom}(f) \setminus \{P\}$ such that $\lim_k \mathbf{x}_k = P$. This is of course true, take for instance, the sequence

$$(1 + 1/k, -1).$$

Now, let's compute the limit. To compute the limit it suffices to compute the limit of each component:

$$\lim_{(x,y) \rightarrow (1,-1)} \frac{x-1}{y-1} = \frac{1-1}{-1-1} = 0$$

since the function $\frac{x-1}{y-1}$ is rational (and hence continuous) and P belongs to its domain.

For the second component (which depends only on x) we get:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,-1)} \frac{x^2+x-2}{x-1} &= \lim_{x \rightarrow 1} \frac{x^2+x-2}{x-1} = \\ \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{x-1} &= \lim_{x \rightarrow 1} x+2 = 3. \end{aligned}$$

Hence

$$\lim_{(x,y) \rightarrow (1,-1)} f(x, y) = (0, 3).$$

§8.3, # 2(a). [10 points] Compute the iterated limits of

$$f(x, y) = \frac{\sin(x) \sin(y)}{x^2 + y^2}.$$

Decide if the function has a limit as $(x, y) \rightarrow (0, 0)$ and prove that the limit exists (or does not exist).

Solution. The iterated limits are:

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{\sin(x) \sin(y)}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0,$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{\sin(x) \sin(y)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0.$$

Hence both iterated limits are equal to zero. Now consider the limit along the line $x = y$:

$$\lim_{x \rightarrow 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin^2(x)}{x^2}.$$

Recall that $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$. Hence by the product theorem for the limits we get:

$$1 = \lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right)^2 = \lim_{x \rightarrow 0} \frac{\sin^2(x)}{x^2}.$$

Thus

$$\lim_{x \rightarrow 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2} \neq 0.$$

Hence the limit along the line $x = y$ is different from the iterated limits, so the limit of the function does not exist. \square

2. [10 points] Compute the iterated limits at $(0, 0)$ of the following function: $f(x, y) = \frac{x^2 + y^4}{x^2 + 2y^4}$. Determine if the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^4}{x^2 + 2y^4}$$

exists.

Solution.

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x^2 + y^4}{x^2 + 2y^4} = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = 1/2.$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^2 + y^4}{x^2 + 2y^4} = \lim_{y \rightarrow 0} \frac{x^2}{x^2} = 1.$$

Hence the iterated limits are distinct, thus the limit does not exist.

3. §8.4, # 1. [10 points] Let $f(x, y) = (x^2, y^2)$. Using definition 8.31 prove that f is differentiable on \mathbb{R}^2 and its total derivative is given by

$$Df(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}.$$

Proof. Let $(a, b) \in \mathbb{R}^2$. Let $\mathbf{h} = (x, y)$. Then to show that

$$Df(a, b) = T = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$$

we have to verify that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

First, consider the numerator:

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h}) = \begin{bmatrix} (a+x)^2 \\ (b+y)^2 \end{bmatrix} - \begin{bmatrix} a^2 \\ b^2 \end{bmatrix} - \begin{bmatrix} 2ax \\ 2ab \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}.$$

Note that $x^4 + y^4 \leq x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$. Thus

$$\|(x^2, y^2)\| \leq x^2 + y^2 = \|(x, y)\|^2 = \|\mathbf{h}\|^2.$$

Hence

$$0 \leq \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} \leq \frac{\|\mathbf{h}\|^2}{\|\mathbf{h}\|} = \|\mathbf{h}\|.$$

Thus by the squeeze theorem,

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \rightarrow \mathbf{0}} \|\mathbf{h}\| = 0. \quad \square$$