MATHEMATICS 3220. Homework # 12: Solutions.

1. §11.5, # 4. Find conditions on a point (x_0, y_0, u_0, v_0) such that there exist functions $u(x,y)$ and $v(x,y)$ which are continuously differentiable near (x_0, y_0) and satisfy

$$\begin{cases} xu^2 + yv^2 + xy = 9 \\ xv^2 + yu^2 - xy = 7 \end{cases}$$

Prove that the solutions satisfy

$$u^2 + v^2 = \frac{16}{x+y}.$$

Solution. First of all, note that by adding these equations we get:

$$(x+y)u^2 + (x+y)v^2 = 16.$$

Hence we get:

$$u^2 + v^2 = \frac{16}{x+y}.$$

Note that $x + y \neq 0$, otherwise we would get: $0 = 16$. Hence it is legal to divide by $x + y$.

There are two ways to continue from here. The first way is to use linear algebra. Our system of equations can be written as

$$\begin{bmatrix} x & y \\ y & x \end{bmatrix} \begin{bmatrix} u^2 \\ v^2 \end{bmatrix} = \begin{bmatrix} 9 - xy \\ 7 + xy \end{bmatrix}.$$

The determinant of the 2-by-2 matrix equals $x^2 - y^2$. Hence, if $|x| \neq |y|$ we get:

$$\begin{bmatrix} u^2 \\ v^2 \end{bmatrix} = \frac{1}{x^2 - y^2} \begin{bmatrix} x & -y \\ -y & x \end{bmatrix} \begin{bmatrix} 9 - xy \\ 7 + xy \end{bmatrix}$$

Hence, if (x_0, y_0, u_0, v_0) satisfies:

$$u_0 > 0, v_0 > 0, |x_0| \neq |y_0|,$$

then (by taking square roots) we can find u and v as continuously differentiable functions of (x, y).

Another way to find u, v is to use the implicit function theorem for the function $F = (F_1, F_2) : \mathbb{R}^4 \to \mathbb{R}^2$,

$$F_1 = xu^2 + yv^2 + xy, \quad F_2 = xv^2 + yu^2 - xy.$$

The conditions of the IMFT are:

$$0 \neq \frac{\partial(F_1, F_2)}{\partial(u, v)} = \begin{vmatrix} 2xu & 2yv \\ 2yu & 2xv \end{vmatrix} = 4uv(x^2 - y^2).$$

To satisfy these conditions we need:

$$|x_0| \neq |y_0|, u_0 \neq 0, v_0 \neq 0.$$

(Compare these with the conditions obtained using the first solution!) Moreover, we would need (x_0, y_0, u_0, v_0) to satisfy the system of equations

$$\begin{cases} x_0 u_0^2 + y_0 v_0^2 + x_0 y_0 = 9 \\ x_0 v_0^2 + y_0 u_0^2 - x_0 y_0 = 7 \end{cases}$$

Under all these conditions we can find the required functions $u(x, y)$ and $v(x, y)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function such that for each $\mathbf{x} \in E = \{(x, y) : f(x, y) = 0\}$ we have $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Show that for each $\mathbf{x}_0 \in E$ there exists a relatively open subset $\Gamma \subset E$ containing $\mathbf{x}_0$ so that either Γ is the graph of a continuously differentiable function $g(x)$ or the graph of a continuously differentiable function $g(y)$.

Solution. Since $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$, at the point $\mathbf{x}_0$ we have:

Either $f_x(x_0, y_0) \neq 0$ or $f_y(x_0, y_0) \neq 0$.

I will consider the first case since the second case is similar. We are in position to apply the implicit function theorem and find a (continuously differentiable) function $g(y)$ defined for $y \in V$ (where $V \subset \mathbb{R}$ is a nonempty open set containing y_0) such that each point $(x, y) = (g(y), y)$ satisfies $f(x, y) = 0$, i.e. such point belongs to E. The set

$$\Gamma = \{(g(y), y) : y \in V\} \subset E$$

is the graph of the function g. It remain to verify that Γ is relatively open in E.

By the uniqueness part of IMFT, for each $y \in V$ the point $(x, y) = (x, g(y))$ is the unique point satisfying $f(x, y) = 0$. Hence if $\mathbf{p}_n \in E$ is a sequence converging to $\mathbf{x} = (x, y) \in \Gamma$ then for all large values of n, $\mathbf{p}_n = (p_n, q_n)$ satisfies

$$p_n \in V, q_n = g(y_n)$$

Hence $\mathbf{p}_n \in \Gamma$ for all large values of n. Thus Γ is relatively open in E. $\square$