MATHEMATICS 3220. Homework # 11: Solutions.

§11.4, # 7. [10 points]. Suppose that V is open in $\mathbb{R}^n$, $H \subset V$ is compact convex subset; $f : V \to \mathbb{R}$ is in $C^2(V)$ and $\nabla f(\mathbf{a}) = 0$ for some $\mathbf{a} \in H$. Prove that there exists a constant M such that

$$|f(\mathbf{x}) - f(\mathbf{a})| \leq M\|\mathbf{x} - \mathbf{a}\|^2,$$

for all $\mathbf{x} \in H$.

Note that the problem was misstated in the textbook, the assumption $\mathbf{a} \in H$ was omitted. Alternatively, one can assume that V is convex.

Solution. Recall that we have Taylor's formula:

$$f(\mathbf{x}) - f(\mathbf{a}) = D^{(1)}f(\mathbf{a}; \mathbf{h}) + \frac{1}{2}D^{(2)}f(\mathbf{c}; \mathbf{h}), \mathbf{h} = \mathbf{x} - \mathbf{a},$$

for some $\mathbf{c}$ in the segment $[\mathbf{a}, \mathbf{x}] \subset H$. Since $\nabla f(\mathbf{a}) = 0$, $D^{(1)}f(\mathbf{a}; \mathbf{h}) = \nabla f(\mathbf{a}) \cdot \mathbf{h} = 0$. Thus we get:

$$f(\mathbf{x}) - f(\mathbf{a}) = \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{c})h_i h_j.$$

Therefore

$$|f(\mathbf{x}) - f(\mathbf{a})| \leq \frac{1}{2}\sum_{i,j=1}^{n} |f_{x_i x_j}(\mathbf{c})h_i h_j|.$$

Note that $|h_i| \leq \|\mathbf{h}\|$ for each $i = 1, ..., n$; hence

$$\sum_{i,j=1}^{n} |\frac{1}{2}f_{x_i x_j}(\mathbf{c})h_i h_j| \leq \sum_{i,j=1}^{n} \frac{1}{2}|f_{x_i x_j}(\mathbf{c})| \cdot \|\mathbf{h}\|^2.$$

Since $f \in C^2(V)$, each function $\frac{1}{2}f_{x_i x_j}$ is continuous on V, hence it is bounded on the compact set H by a constant M. Thus

$$|f(\mathbf{x}) - f(\mathbf{a})| \leq M\|\mathbf{x} - \mathbf{a}\|^2. \quad \square$$

§11.5, # 1(b). [10 points]. Prove that for the following function f the inverse function f^{-1} exists and is differentiable on some nonempty open set containing $\mathbf{a}$ and compute $Df^{-1}(\mathbf{a})$:

$$f(u, v) = (u + v, \sin(u) + \cos(v)), \mathbf{a} = (0, 1).$$

Solution. First, consider the equation $f(u, v) = (0, 1)$:

$$u + v = 0, \quad \sin(u) + \cos(v) = 1,$$

$$v = -u, \quad \sin(u) + \cos(u) = 1$$

The equation $\sin(u) + \cos(u) = 1$ has solutions $t_n = 2\pi n$ and $u_n = 2\pi n + \frac{\pi}{2}$. (One can actually see that these are the only solutions, although we do not need this: by squaring the equation $\sin(u) + \cos(u) = 1$ we get:

$$\sin^2(u) + \cos^2(u) + 2\sin(u)\cos(u) = 1,$$

$$\sin(u)\cos(u) = 0.$$

Hence either $\sin(u) = 0$ or $\cos(u) = 0$. In the former case $\cos(u) = 1$, in the second case $\sin(u) = 1$. This leaves us with the solutions $t_n = 2\pi n$ and $u_n = 2\pi n + \frac{\pi}{2}$).

Thus we get solutions $\mathbf{p}_n = (t_n, s_n) = (2\pi n, -2\pi n)$ and $\mathbf{q}_n = (u_n, v_n) = (2\pi n + \frac{\pi}{2}, -2\pi n - \frac{\pi}{2})$; $f(\mathbf{p}_n) = \mathbf{a}$, $f(\mathbf{q}_n) = \mathbf{a}$.

Let's compute $Df(\mathbf{p}_n)$:

$$Df(\mathbf{p}_n) = \begin{bmatrix} 1 & 1 \\ \cos(t_n) & -\sin(s_n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The determinant of this matrix equals -1, hence it is invertible and the inverse matrix equals

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

This is sufficient to solve the problem: by the inverse function theorem there exists an inverse function f^{-1} defined on an open ball around $\mathbf{a}$ such that $f^{-1}(\mathbf{a}) = (0,0)$ (here I am taking $n = 0$ but you can take any integer n you like). The total derivative at $\mathbf{a}$ of this inverse function equals

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

For the sake of completeness let's consider the other sequence of solutions: $\mathbf{q}_n = (2\pi n + \frac{\pi}{2}, -2\pi n - \frac{\pi}{2})$. Then

$$Df(\mathbf{q}_n) = \begin{bmatrix} 1 & 1 \\ \cos(u_n) & -\sin(v_n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The determinant of this matrix is 1 and its inverse is

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Thus by the inverse function theorem there exists an inverse function f^{-1} defined on an open ball around $\mathbf{a}$ such that $f^{-1}(\mathbf{a}) = (\frac{\pi}{2}, -\frac{\pi}{2})$ (here I am taking $n = 0$ but you can take any integer n you like). The total derivative at $\mathbf{a}$ of this inverse function equals

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Note that both answers A and B for the total derivative of inverse function at $\mathbf{a}$ are correct.

§11.5, # 1(c). [10 points]. Prove that for the following function f the inverse function f^{-1} exists and is differentiable on some nonempty open set containing $\mathbf{a}$ and compute $Df^{-1}(\mathbf{a})$:

$$f(u,v) = (uv, u^2 + v^2), \mathbf{a} = (2,5).$$

Solution. We again start by solving the equation $f(u,v) = (2,5)$:

$$uv = 2, u^2 + v^2 = 5.$$

There are four obvious solutions of this equation: $(u,v) = (1,2),(-1,-2),(2,1),(-2,-1)$. To see that there are no other solutions make the substitution $v = 2/u$ and get $u^2 + \frac{4}{u^2} = 5$, or

$$u^4 + 4 = 5u^2.$$

This is an equation of order 4, hence it cannot have more than 4 solutions, thus we have found all the solutions. Now let's compute the total derivative:

$$Df(u,v) = \begin{bmatrix} v & u \\ 2u & 2v \end{bmatrix}.$$

Hence

$$A_1 = Df(1,2) = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}, \quad A_2 = Df(-1,-2) = \begin{bmatrix} -2 & -1 \\ -2 & -4 \end{bmatrix},$$

$$A_3 = Df(2,1) = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad A_4 = Df(-1,-2) = \begin{bmatrix} -1 & -2 \\ -4 & -2 \end{bmatrix}.$$

The determinants of these matrices are equal to $6, 6, -6, -6$ respectively. Hence the total derivatives are invertible for all points $\mathbf{p}_1 = (1,2), \mathbf{p}_2 = (-1,-2), \mathbf{p}_3 = (2,1), \mathbf{p}_4 = (-2,-1)$. Thus by the inverse function theorem there are functions g_1, g_2, g_3, g_4 inverse to f and defined on a small ball around $\mathbf{a}$ so that $g_i(\mathbf{a}) = \mathbf{p}_i$, each g_i is differentiable at $\mathbf{a}$ and

$$Dg_1(\mathbf{a}) = A_1^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -1 \\ -2 & 2 \end{bmatrix},$$

$$Dg_2(\mathbf{a}) = A_2^{-1} = \frac{1}{6}\begin{bmatrix} -4 & 1 \\ 2 & -2 \end{bmatrix},$$

$$Dg_3(\mathbf{a}) = A_3^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 2 \\ 4 & -1 \end{bmatrix},$$

$$Dg_4(\mathbf{a}) = A_4^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ -4 & 1 \end{bmatrix}.$$