MATHEMATICS 3220. Homework # 1: Solutions.

§8.1, # 10. (a) [20 points] Prove that ℓ_1-norm and the sup-norm are norms, i.e. they satisfy:

(1) $\|\mathbf{x}\| \geq 0$ for each $\mathbf{x} \in \mathbb{R}^n$ with equality if and only if $\mathbf{x} = 0$.

(2) $\|\alpha\mathbf{x}\| = |\alpha| \cdot \|\mathbf{x}\|$ for each $\alpha \in \mathbb{R}$.

(3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof. First, consider the ℓ_1-norm, $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$. Since $|x_i| \geq 0$ for each i, the sum of these numbers would be also nonnegative, hence $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| \geq 0$. If $\mathbf{x} = 0$ then clearly $\|\mathbf{x}\|_1 = 0$. If $\mathbf{x} \neq 0$ then there is a coordinate, say, x_i such that $|x_i| > 0$. Hence

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| \geq |x_i| > 0.$$

This proves (1). For each $\alpha \in \mathbb{R}$ we have:

$$\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^n |\alpha x_i| = \sum_{i=1}^n |\alpha||x_i| = |\alpha| \sum_{i=1}^n |x_i| = |\alpha| \cdot \|\mathbf{x}\|_1.$$

This proves (2). By the triangle inequality in $\mathbb{R}$ we have:

$$\|\mathbf{x} + \mathbf{y}\|_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1.$$

This proves (3).

Now consider the sup-norm, $\|\mathbf{x}\|_\infty = \max\{|x_i|, i = 1, ..., n\}$. Since $|x_i| \geq 0$ for each i, $\max\{|x_i|, i = 1, ..., n\} \geq 0$. If $\mathbf{x} = 0$ then $|x_i| = 0$ for each i, hence $\max\{|x_i|, i = 1, ..., n\} = 0$. If $\mathbf{x} \neq 0$, then, say, $|x_i| \neq 0$. Thus

$$\|\mathbf{x}\|_\infty = \max\{|x_i|, i = 1, ..., n\} \geq |x_i| > 0.$$

This proves (1). For each $\alpha \in \mathbb{R}$ we have:

$$\|\alpha\mathbf{x}\|_\infty = \max\{|\alpha| \cdot |x_i|, i = 1, ..., n\};$$

$$\|\mathbf{x}\|_\infty = \max\{|x_i|, i = 1, ..., n\}$$

The maximum in the latter is achieved on, say, $i = k$, i.e.

$$\|\mathbf{x}\|_\infty = |x_k|.$$

Thus $|\alpha| \cdot |x_k| \geq |\alpha| \cdot |x_i|$ for each i, hence

$$|\alpha| \cdot \|\mathbf{x}\|_\infty = |\alpha| \cdot |x_k| = \max\{|\alpha| \cdot |x_i|, i = 1, ..., n\} = \|\alpha\mathbf{x}\|_\infty.$$

This proves (2). Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}\|_\infty = |x_k|$, $\|\mathbf{y}\|_\infty = |y_m|$. Then for each i we have:

$$|x_i + y_i| \leq |x_i| + |y_i| \leq |x_k| + |y_m| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

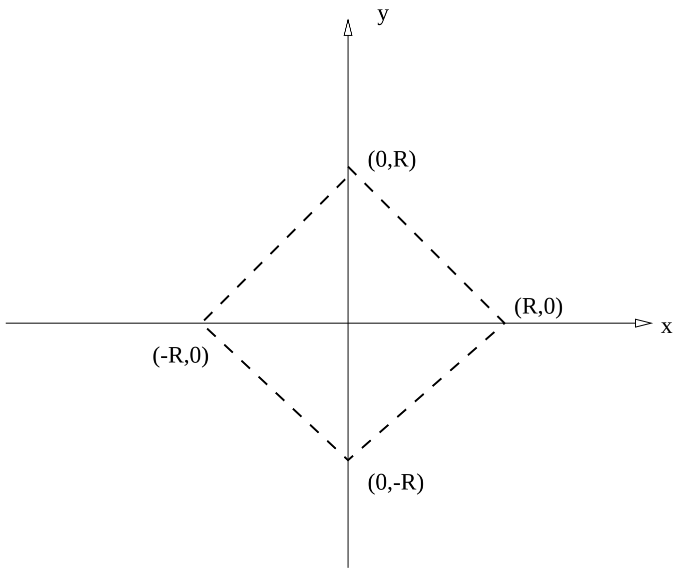

Figure 1:

Thus

$$\|\mathbf{x}+\mathbf{y}\|_\infty = \max\{|x_i+y_i|, i=1,...,n\} \le \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

This proves (3).

(b) [20 points] Describe what the open balls in $\mathbb{R}^2$ look like with respect to the nonEuclidean norms $\|\cdot\|_1, \|\cdot\|_\infty$.

Solution. We will describe the open ball of radius $R > 0$ centered at the origin,

$$B_R(0) = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_i < R\}$$

$i = 1, i = \infty$. First we consider the ℓ_1-norm.

$$B_R(0) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < R\}.$$

We decribe the ball $B_R(0)$ in each of the coordinate quadrants in $\mathbb{R}^2$. In the first quadrant $(x \ge 0, y \ge 0)$ the inequality $|x|+|y| < R$ becomes

$$x + y < R \iff y < R - x$$

which decribes the region in the first quadrant that is contained below the line $y = R - x$ (this is a line segment with the slope -1 connecting the points $(R,0)$ and $(0,R)$). Similarly, in the second quadrant, $(x \le 0, y \ge 0)$ the inequality $|x| + |y| < R$ becomes

$$-x + y < R \iff y < R + x.$$

This describes the triangle in the second quadrant which is contained below the line $y = R + y$. Similarly, in the third quadrant $(x \le 0, y \le 0)$ we get the triangle above the line $y = -x - R$; in the fourth quadrant $(x \ge 0, y \le 0)$ we get the triangle above the line

$y = x - R$. Combining these answers we get the square with the vertices $(R, 0), (0, R), (-R, 0), (0, -R)$, see Figure 1.

Now consider the sup-norm. The open ball of radius $R > 0$ centered at the origin is

$$B_R(0) = \{(x, y) \in \mathbb{R}^2 : \max(|x|, |y|) < R\}.$$

Saying that $\max(|x|, |y|) < R$ is the same as to say $|x| < R$ and $|y| < R$. We again consider one coordinate quadrant at a time. In the first quadrant we get the region

$$\{(x, y) : 0 \leq x, 0 \leq y, \max(x, y) < R\} = \{(x, y) : 0 \leq x < R, 0 \leq y < R\}.$$

The latter is the square with the vertices $(0, 0), (R, 0), (R, R), (0, R)$. Similarly, we get R-by-R-squares in each coordinate quadrant. Hence $B_R(0)$ in the sup-norm is the square with the vertices $(R, R), (-R, R), (-R, -R), (R, -R)$, see Figure 2.

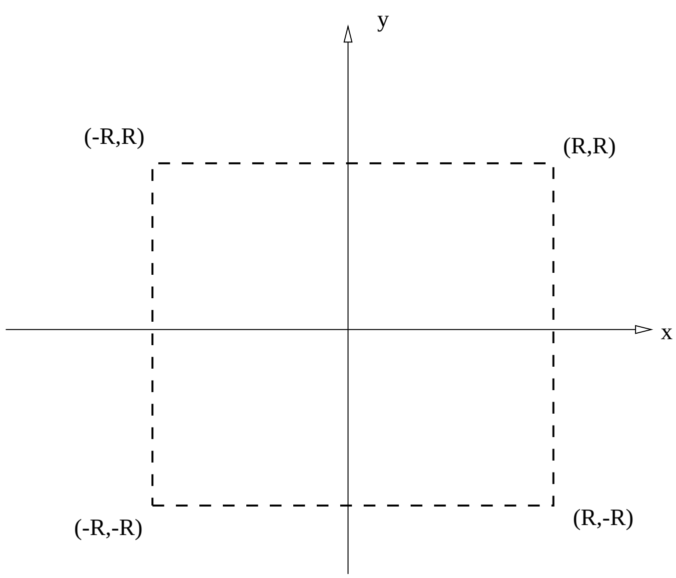

Figure 2:

The balls with centers at other points $P = (a, b)$ of $\mathbb{R}^2$ are obtained by parallel translation (along the vector (a, b)) of the balls with the center at $(0, 0)$, since

$$\mathbf{v} \in B_R(0) \iff \|(\mathbf{v} + (a, b)) - (a, b)\|_i < R \iff \mathbf{v} + (a, b) \in B_R(0).$$

Hence these balls are again squares with the vertices:

$$(a + R, b), (a, b + R), (a - R, b), (a, b - R)$$

(for the ℓ_1-norm) and with the vertices:

$$(a + R, b + R), (a + R, b - R), (a - R, b - R), (a - R, b - R)$$

(for the sup-norm).