MATHEMATICS 3220. Homework # 8.

1. Rearrange the following sentances to get a valid proof of the following

Theorem. If H is a compact set and $f : H \to \mathbb{R}^m$ is a continuous function then f is uniformly continuous.

Proof. The definition of uniform continuity is: For each $\epsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}, \mathbf{x}' \in H$ we have

$$\|\mathbf{x} - \mathbf{x}'\| < \delta \Rightarrow \|f(\mathbf{x}) - f(\mathbf{x}')\| < \epsilon.$$

Thus the sequence $\|f(\mathbf{x}_{n_j}) - f(\mathbf{x}'_{n_j})\|$ cannot converge to zero. Since H is compact, there is a subsequence $(\mathbf{x}_{n_j})_{j\in\mathbb{N}}$ which converges to some $\mathbf{x} \in H$. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ there are $\mathbf{x}, \mathbf{x}' \in H$ so that

$$\|\mathbf{x} - \mathbf{x}'\| < \delta \text{ and } \|f(\mathbf{x}) - f(\mathbf{x}')\| \geq \epsilon.$$

Since f is continuous at $\mathbf{x} \in H$ we get:

$$\lim_{j\to\infty} f(\mathbf{x}_{n_j}) = f(x) = \lim_{j\to\infty} f(\mathbf{x}'_{n_j}).$$

Then for each $n \in \mathbb{N}$ there are $\mathbf{x}_n, \mathbf{x}'_n \in H$ such that

$$\|\mathbf{x}_n - \mathbf{x}'_n\| < 1/n \text{ and } \|f(\mathbf{x}_n) - f(\mathbf{x}'_n)\| \geq \epsilon.$$

However, $\|f(\mathbf{x}_{n_j}) - f(\mathbf{x}'_{n_j})\| \geq \epsilon > 0$ for each j. By the triangle inequality we have:

$$0 \leq \|\mathbf{x} - \mathbf{x}'_n\| \leq \|\mathbf{x} - \mathbf{x}_n\| + \|\mathbf{x}_n - \mathbf{x}'_n\| < \|\mathbf{x} - \mathbf{x}_n\| + 1/n.$$

Suppose that uniform continuity fails. Since $\lim_{j\to\infty} \|\mathbf{x} - \mathbf{x}_{n_j}\| = 0$, by applying squeeze lemma we get:

$$\lim_{n\to\infty} \|\mathbf{x} - \mathbf{x}'_{n_j}\| = 0 \Rightarrow \lim_{n\to\infty} \mathbf{x}'_{n_j} = \mathbf{x}.$$

Hence

$$\lim_{j\to\infty} \|f(\mathbf{x}_{n_j}) - f(\mathbf{x}'_{n_j})\| = 0.$$

Let's choose δ of the form $1/n$. Contradiction. Hence f is uniformly continuous on H. □

§11.2, # 3: Prove that the function $\sqrt{|xy|}$ is not differentiable at $(0,0)$.

Solution. Suppose that f is differentiable