MATH 2270-2. Final Test: Solutions.

Problem 1. (10 points) Find inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Use any method you like.

Solution.

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}. \quad \square$$

Problem 2. (15 points) Let V be the linear space of continuous functions of one variable $f : \mathbb{R} \to \mathbb{R}$ (with the usual operations of sum and multiplication by scalars). Let Z be the subset in V which consists of all continuous functions f which have integer values at zero (i.e., we allow $f(0)$ to be $0, \pm 1, \pm 2, ...$, but for instance $f(0) = 0.5$ is not allowed). Determine whether or not the subset Z is a subspace in V. Justify your answer!

Solution. Z is not a subspace. For instance, take function $f(x) = 1$, $\alpha = 0.5$. Then $\alpha f(0) = 0.5$ is not an integer, i.e. $\alpha f(x)$ does not belong to Z. $\square$

Problem 3. (15 points) 3. [15 points] Using standard basis in P_2 find matrix representation, rank, nullity and basis of the image of the linear transformation $T : P_2 \to P_2$ which is given by the formula:

$$T(p(x)) = xp'(x) + x^2 p(1).$$

Here p' denotes the derivative.

Solution. $T(1) = 0 + x^2$, has coordinates $(0, 0, 1)$. Next, $T(x) = x + x^2$, has coordinates $(0, 1, 1)$. Lastly, $T(x^2) = 2x^2 + x^2 = 3x^2$ has coordinates $(0, 0, 3)$. Hence the matrix of T is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}.$$

This matrix has rank 2 and nullity 1. The basis of the image consists of the vectors $(0, 0, 1)$ and $(0, 1, 1)$. Converting these vectors into polynomials we get: basis of the image of T is $\{x^2, x + x^2\}$. $\square$

Problem 4. (10 points) Write the augmented matrix and find all solutions of the linear system:

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 3 \\ 2x_1 + x_2 + 3x_3 + 2x_4 = 5 \\ 3x_1 + 2x_2 + 4x_3 = 8 \end{cases}$$

Solution. $x_1 = 2 - 2t - 4s$, $x_2 = 1 + t + 6s$, $x_3 = t, x_4 = s$ are the parameters. $\square$

Problem 5. (15 points) Find an orthonormal basis in the subspace V in $\mathbb{R}^4$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -4 \\ 6 \end{pmatrix}.$$

Solution. $\overrightarrow{u}_1 = \frac{1}{2}(1, 1, -1, 1)$, $\overrightarrow{u}_2 = \frac{1}{2}(-1, 1, 1, 1)$. $\square$

Problem 6. (15 points) Find all eigenvalues of the matrix:

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

For each eigenvalue find a basis of the corresponding eigenspace. Determine if the matrix is diagonalizable.

Solution. The first eigenvalue is 3, the basis of E_3 is $(0, 0, 1)$. The second eigenvalue is 2, the basis of E_2 is $(0, -1, 1)$. Matrix is not diagonalizable since the sum of dimensions of eigenspaces is 2 and not 3. $\square$

Problem 7. (10 points) Compute the following determinant using the definition of the determinant (i.e. identify the patterns with the nonzero product of the entries, compute the $\pm$ signs, etc.):

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

Solution. There is only one pattern with nonzero product, it is the product of the entries equal to 1. The pattern contains 2 inversions, hence the determinant equals $(-1)^2 = 1$. $\square$

Problem 8. (10 points) Consider the vector space

$$V = Span\{\sin^2(t), \cos^2(t), t\}$$

with the basis: $T = \{\sin^2(t), 1, t - 1\}$. Compute the coordinates of the function $2(\sin^2(t) + \cos^2(t)) - t$ with respect to the basis T.

Solution. $2(\sin^2(t) + \cos^2(t)) - t = 0 \cdot \sin^2(t) + 1 \cdot 1 + (-1) \cdot (t - 1)$. Hence the coordinates are $(0, 1, -1)$. $\square$