MATHEMATICS 2270-1. Second Midterm Test: Solutions.

Problem 1. [15 points] Suppose that A is an invertible matrix $n \times n$ with the eigenvalues $\lambda_1, ..., \lambda_k$. What are eigenvalues of A^{-1}? Justify your answer.

Solution. First of all, since A is invertible, all its eigenvalues are nonzero. For each i there exists a nonzero vector v_i such that

$$Av_i = \lambda_i v_i.$$

By multiplying this equation by A^{-1} and by λ_i^{-1} we get:

$$\frac{1}{\lambda_i} v_i = A^{-1} v_i.$$

Hence $\frac{1}{\lambda_i}$ is an eigenvalue of A^{-1} for each i. The same logic shows that these are the only eigenvalues of A^{-1}. Hence the eigenvalues of A^{-1} are $\lambda_1^{-1}, ..., \lambda_k^{-1}$. □

Problem 2. (20 points) Compute eigenvalues and bases of the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

Using this, compute the limit $\lim_{n \to \infty} A^n$.

Solution.

$$A = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

Let's diagonalize the matrix

$$B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

The eigenvalues of B are the roots of the polynomial $(\lambda - 2)(\lambda - 3) - 2 = \lambda^2 - 5\lambda + 4$. Hence the eigenvalues of B are $1, 4$. Thus the eigenvalues of A are $1/4, 1$. Now let's find the eigenvectors of B:

E_1 is the kernel of

$$\begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix},$$

thus the basis of E_1 is the vector $v_1 = (1, -1)$.

E_4 is the kernel of

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix},$$

thus the basis of E_4 is the vector $v_1 = (1, 2)$. Hence the eigenbasis of B (and hence of A) is:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, S^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix},$$

hence

$$A = \frac{1}{4}B = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Hence

$$A^n = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1/(4^n) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

The limit $\lim_{n\to\infty} 1/(4^n)$ is zero. Hence

$$\lim_{n\to\infty} A^n = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}. \quad \square$$

Problem 3. [15 points] Using standard basis in P_2 find matrix representation, and eigenvalues of the linear transformation $T : P_2 \to P_2$ which is given by the formula:

$$T(p(x)) = p(x+1) - p'(x).$$

Solution. $T(1) = 1 - (1)' = 1$, has coordinates $(1, 0, 0)$. $T(x) = x + 1$, has coordinates $(1, 0, 0)$. $T(x^2) = (x+1)^2 - (x^2)' = x^2 + 2x + 1 - 2x = 1 + x^2$, the latter has coordinates $(1, 0, 1)$. Thus the matrix of T is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix is upper triangular, hence its eigenvalues are its diagonal entries. Thus A has only one eigenvalue: $\lambda = 1$. $\square$

Problem 4. [15 points] Determine if the quadratic form q is positive definite: $q(x, y) = x^2 + 3y^2 + 4xy$. Justify your anwer.

Solution. The matrix of this quadratic form is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

The determinant of this matrix is $3 - 4 = -1 < 0$, hence A is not positive-definite. Therefore q is not positive definite. $\square$

Problem 5. [15 points] Is the following set S a linear subspace in P_2? Justify your answer. Find a basis of S if S is linear.

$$S = \{p(x) : p(0) = p(1)\}.$$

Solution. First, let's check that S is a subspace:

(a) $p(x) = 0$ is in S, since $0 = p(0) = p(1) = 0$.

(b) Suppose that $p(x), q(x)$ are in S. Thus $p(0) = p(1), q(0) = q(1)$. It follows that $(p+q)(0) = p(0) + q(0) = p(1) + q(1) = (p+q)(1)$. Thus $p + q$ is in S.

(c) Suppose that α is a scalar, $p(x)$ is a polynomial in S. Thus $p(0) = p(1)$. It follows that $\alpha p(0) = \alpha p(1)$. Hence $\alpha p(x)$ is in S.

Thereore S is a subspace.

Second, let's find the general polynomial in S:

$$p(x) = a_0 + a_1 x + a_2 x^2, p(0) = p(1) \Rightarrow$$

$$a_0 = a_0 + a_1 + a_2,$$

hence $a_2 = -a_1$, and a_0, a_1 are arbitrary parameters. Thus to find a basis we first let $a_0 = 1, a_1 = 0$, hence we get $p(x) = 1$. Next, we take $a_0 = 0, a_1 = 1$, hence we get $q(x) = x - x^2$. Therefore $\{1, x - x^2\}$ is a basis. □

Problem 6. (20 Points) Determine if the following matrices are similar:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Justify your anwer.

Solution. Note that A and B have same eigenvalues of the same *algebraic* multiplicity: 1 is an eigenvalue of algebraic multiplicity 2 and 2 is an eigenvalue of algebraic multiplicity 1. However this is not enough for similarity of the matrices. Let's compute *geometric multiplicities* of the eigenvalue 1.

For the matrix A the subspace E_1 is the kernel of the matrix

$$I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The latter has rank 2 and nullity 1. Therefore the geometric multiplicity of 1 in this case equals $dim(E_1) = 1$.

For the matrix B the subspace E_1 is the kernel of the matrix

$$I - B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The latter has rank 1 and nullity 2. Therefore the geometric multiplicity of 1 in this case equals $dim(E_1) = 2$. For similar matrices geometric multiplicities of eigenvalues should be the same, therefore A, B are not similar.

Alternative solution. Check that B is diagonalizable and A is not. Therefore these matrices cannot be similar. □