MATHEMATICS 2270-1. First Midterm Test: Solutions.

1. [15 points] Find all solutions of the linear system (in 5 unknowns):

$$\begin{cases} x_2 + x_4 + x_5 = 1 \\ x_1 + x_3 = 0 \\ x_1 + x_2 + x_5 = 1 \end{cases}$$

Solution. Applying Gauss-Jordan method to the augmented matrix of this system we get:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = t, x_2 = 1 - t - s, x_3 = -t, x_4 = t, x_5 = s$, where t, s are arbitrary real numbers. □

2. [15 points] Using the row echelon reduction find inverse of the matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution. Applying Gauss-Jordan method to the augmented matrix we get:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$$

Therefore the inverse matrix equals

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}. \quad \square$$

3. [15 points] Find coordinates of the vector $\vec{v} = (4, 4)$ with respect to the basis

$$\mathcal{B} = \{\begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}.$$

Solution. Form the transition matrix

$$S = \begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}.$$

The inverse of this matrix equals

$$S^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ -6 & 5 \end{bmatrix}.$$

Therefore the coordinates of the vector $\overrightarrow{v}$ with respect to the basis $\mathcal{B}$ equal to

$$[\overrightarrow{v}]_{\mathcal{B}} = S^{-1}\overrightarrow{v} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence

$$[\overrightarrow{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad \square$$

4. [20 points] Find rank, nullity and a basis of the kernel of the linear transformation

$$T(\overrightarrow{x}) = \begin{pmatrix} x+y+z \\ x+y+z \\ 2x+2y+2z \end{pmatrix}, \quad \text{where } \overrightarrow{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution. The matrix of T equals

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

Applying Gauss-Jordan method to this matrix we get:

$$A \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A_{rref}.$$

Hence $rank(A) = 1$, nullity of A equals $3 - 1 = 2$. Finally, let's find a basis of the kernel. The kernel of A is the same as the kernel of A_{rref}, equals

$$(x, y, z) : x = -y - z$$

where y, z are the parameters. Thus to get a basis we first take $y = 1, z = 0$, and then $y = 0, z = 1$; this yields a basis of the kernel:

$$\overrightarrow{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \quad \square$$

5. [20 points] Decide if the following set forms a subspace in $\mathbb{R}^2$. Justify your answer.

$$S = \{(x, y) : y \geq x^2\}.$$

(It could help you to draw a picture of this set before trying to solve the problem.)

Solution. The above set consists of points which lie above the parabola $y = x^2$ in the xy-plane. Hence to see that S is not a subspace we can take the vector

$$\overrightarrow{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

it belongs to S since $1 \geq 0^2 = 0$. Now, take the scalar $\alpha = -1$, then

$$\alpha \overrightarrow{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

this vector does not belong to S because $-1 < 0^2$. Thus S is not a subspace. $\square$

6. [15 points] Compute the following determinant using the definition of the determinant:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

Solution. Let's first try to find all patterns in this determinant which consist of nonzero numbers: the only such pattern is

$$\begin{vmatrix} \square & & & \\ & & & \square \\ & \square & & \\ & & \square & \end{vmatrix}.$$

This pattern contains two inversions, which is an even number. The product of the numbers in this pattern equals 1; therefore the determinant equals 1.