MATHEMATICS 2270-2. Second Midterm Test: Sample (Solutions).

1. [15 points] Use the "classical adjoint" to find inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. $det(A) = 2$, hence

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 \cdot 1 & 0 & -2 \cdot 1 \\ 0 & 1 \cdot 1 & 0 \\ 0 & 0 & 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

2. [15 points] Is the following set a subspace in P_2? Find a basis if it is.

$$S = \{p(t) : p'(0) = p(1)\}.$$

Solution. First, let's check that S is a subspace. Clearly, zero polynomial belongs to S. Suppose that p, q belong to S; thus $p'(0) = p(1)$, $q'(0) = q(1)$. It follows that for $p + q$ we have:

$$(p+q)'(0) = p'(0) + q'(0) = p(1) + q(1) = (p+q)(1).$$

Hence $p + q$ is also in S. The argument for the scalar multiples αp is similar. Now, let's find the general polynomial in S:

$$p(t) = a_0 + a_1 t + a_2 t^2,$$

$$p'(0) = a_1 = p(1) = a_0 + a_1 + a_2,$$

hence $a_0 = -a_2$, and a_1, a_2 are arbitrary numbers. Thus to get basis we first take $a_1 = 1, a_2 = 0$ and therefore $p(t) = t$; then we take $a_1 = 0, a_2 = 1$ and therefore $q(t) = -1 + t^2$. Hence a basis is $\{t, -1 + t^2\}$. $\square$

3. [15 points] Using standard bases in P_1, P_2 find matrix representation, rank and nullity of the linear transformation $T : P_1 \to P_2$ which is given by the formula:

$$T(p(x)) = xp(x)$$

Solution. $T(1) = x$, $T(x) = x^2$, hence the matrix is

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix has rank 2 and nullity 0. $\square$

4. [20 points] Use characteristic polynomial to find eigenvalues of the matrix A and bases of the corresponding eigenvectors:

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}.$$

Diagonalize the matrix A if possible. If it is impossible, explain why.

Solution. The characteristic polynomial is

$$\begin{vmatrix} \lambda+1 & 1 & 1 \\ 1 & \lambda+1 & 1 \\ 1 & 1 & \lambda+1 \end{vmatrix} = (\lambda+1)^3 + 2 - 3(\lambda+1) = \lambda^3 + 3\lambda = \lambda^2(\lambda+3).$$

Thus the eigenvalues are 0 (of multiplicity 2) and -3. The basis of eigenvectors for the eigenvalue 0 is $(-1,1,0), (-1,0,1)$. The basis for the eigenvalue $\lambda = -3$ is the kernel of

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the eigenvector is $(-3,1,1)$. Thus the basis is

$$(-1,1,0), (-1,0,1), (-3,1,1).$$

Therefore the matrix A is diagonalizable with the diagonal matrix

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$S = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

5. [20 points] Find complex eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Use these eigenvalues to compute A^{990}.

Solution. The characteristic equation is $\lambda^3 = -1$, its solutions are: $\lambda_1 = -1$, $\lambda_2 = e^{i\pi/3}$, $\lambda_3 = e^{-i\pi/3}$. Thus A is similar to the diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & e^{i\pi/3} & 0 \\ 0 & 0 & e^{-i\pi/3} \end{bmatrix},$$

$$A = SDS^{-1},$$

for an appropriate complex matrix S. Therefore

$$A^{990} = SD^{990}S^{-1} = S \begin{bmatrix} (-1)^{990} & 0 & 0 \\ 0 & e^{330i\pi} & 0 \\ 0 & 0 & e^{-i330\pi} \end{bmatrix} S^{-1} =$$

$$S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

6. [15 points] Determine if the following formula defines an inner product on $\mathbb{R}^2$:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y},$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Solution. The matrix A is symmetric, hence the formula defines an inner product if and only if A is positive definite. The determinant of A equals $-2 - 1 = -3$ which is a negative number. Thus A is not positive-definite, therefore the formula does not define an inner product.