MATHEMATICS 2270-2. First Midterm Test (Sample): Solutions.

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1. Find all solutions of the linear system

$$\begin{cases} x_3 + x_4 + x_5 + x_6 = 0 \\ x_1 + x_2 - 2x_6 = 0 \\ x_2 + x_4 - x_5 = 0 \end{cases}$$

Solution. Applying Gauss-Jordan method to the augmented matrix of this system we get:

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

Therefore x_4, x_5, x_6 are parameters, and the solution is:

$$x_1 = x_4 - x_5 - 2x_6, x_2 = -x_4 + x_5, x_3 = -x_4 - x_5 - x_6,$$

x_4, x_5, x_6 are arbitrary real numbers. □

2. Using the row echelon reduction find inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Solution. Applying Gauss-Jordan method to the augmented matrix we get:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$$

Therefore the inverse matrix equals

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

3. Determine if the following vectors are linearly independent:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Solution. Consider the matrix A whose columns are the above vectors and apply Gauss-Jordan algorithm:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & -3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $rank(A) = 2 < 3$, which implies that the vectors are linearly dependent. □

4. Find basis of the image of the linear transformation

$$T(\vec{x}) = \begin{pmatrix} x - y + 2z \\ x - y + 2z \\ x - y + 2z \end{pmatrix}, \quad \text{where } \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution. Apply Gauss-Jordan method to the matrix A of T:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since only the first column contains a leading entry, it follows that the first column of A is a basis of the image:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad \square$$

5. Find an orthonormal basis in the subspace V in $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 6 \\ -4 \\ 6 \end{pmatrix}.$$

Solution. First, let's normalize the vector $\vec{v}_1$:

$$|\vec{v}_1| = \sqrt{1+1+1+1} = 2,$$

therefore

$$\vec{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore the projection of $\vec{v}_2$ to the line L through $\vec{u}_1$ equals

$$proj_L(\vec{v}_2) = (\vec{v}_2 \cdot \vec{u}_1)(\frac{1}{2})\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Hence let $\vec{w}_2 := \vec{v}_2 - proj_L(\vec{v}_2) =$

$$\begin{pmatrix} 4 \\ 6 \\ -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Finally, let's normalize the vector $\vec{w}_2$:

$$\vec{u}_2 := \vec{w}_2/|\vec{w}_2| = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore the orthonormal basis equals:

$$\vec{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \vec{u}_2 = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad \square$$

6. Compute the following determinant using the definition of the 3×3 determinant:

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix}.$$

Solution. The determinant (using for instance the "2 star rule") equals:

$$(1)(2)(0) + (1)(1)(0) + (1)(2)(3) - [(1)(2)(2) + (1)(1)(0) + (1)(3)(0)] = 6 - 4 = 2. \quad \square$$