MATHEMATICS 2270. Homework # 9: Solutions.

Total=60 points.

1. [10 points] Consider the vector space

$$V = Span\{\sin(t), \cos(t)\}$$

with the basis: $\mathcal{B} = \{f(t) = \sin(t) - \cos(t), g(t) = \sin(t) + 2\cos(t)\}$. Compute the coordinates of the function $2\sin(t) - \cos(t)$ with respect to the basis $\mathcal{B}$.

Solution. There are two solutions of this problem.

Solution 1. The coordinates of the function $2\sin(t) - \cos(t)$ with respect to the basis $\mathcal{A} := \{\sin(t), \cos(t)\}$ are $(2, -1)$. The coordinates of the vectors f and g with respect to the basis $\mathcal{A}$ are:

$$[f]_{\mathcal{A}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, [g]_{\mathcal{A}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus the transition matrix from the $\mathcal{A}$–coordinates to the $\mathcal{B}$–coordinates is the inverse of

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix},$$

i.e.

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Hence the coordinates of the function $h(t) = 2\sin(t) - \cos(t)$ with respect to the basis $\mathcal{B}$ are:

$$S^{-1}[h]_{\mathcal{A}} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}.$$

Thus the coordinates are $(5/3, 1/3)$.

Solution 2. To find coordinates of the function h in ine $\mathcal{B}$-coordinates we have to find numbers a and b satisfying the equation

$$h(t) = af(t) + bg(t),$$

i.e.,

$$2\sin(t) - \cos(t) = a(\sin(t) - \cos(t)) + b(\sin(t) + 2\cos(t)) = (a+b)\sin(t) + (2b-a)\cos(t),$$

i.e.

$$\begin{cases} a + b = 2 \\ -a + 2b = -1 \end{cases}$$

Solving this system we get:

$$\begin{cases} a + b = 2 \\ 0 + 3b = 1 \end{cases} \iff \begin{cases} a + b = 2 \\ b = 1/3 \end{cases} \iff \begin{cases} a = 2 - (1/3) = 5/3 \\ b = 1/3 \end{cases}$$

Hence the coordinates are $(5/3, 1/3)$. □

2. [10 points] Determine whether or not the following collection of vectors spans P_2:

$$\{p_1(t) = t^2 + 1,\ p_2(t) = 2t + 1,\ p_3(t) = t^2 + 2t + 2,\ p_4(t) = -t^2 + 2t\}.$$

(Hint: first find dimension of the space $Span\{p_1(t), p_2(t), p_3(t), p_4(t)\}$.)

Solution. Let's write down the matrix A whose columns are the coordinates of the polynomials $p_1, ..., p_4$ with respect to the standard basis:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

Dimension of $Span\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ is the rank of the matrix A. To compute the rank of A let's apply the Gauss-Jordan method:

$$A \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the reduced row eachelon form of A has two nonzero rows, therefore $rank(A) = 2$. Thus dimension of $Span\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ equals 2 while the space P_2 is 3-dimensional. Therefore the vectors $p_1(t), p_2(t), p_3(t), p_4(t)$ do not span P_2. $\square$

3. [10 points] Find matrix representation, rank and nullity of the linear transformation $T : P_1 \to P_2$ which is given by the formula:

$$T(p(x)) = xp(x)$$

Here P_1 is the space of polynomials $b_0 + b_1 x$ of degree ≤ 1 (with the standard basis $\{1, x\}$) and P_2 is the space of polynomials $a_0 + a_1 x + a_2 x^2$ of the degree at most 2 with the standard basis $\{1, x, x^2\}$.

Solution. The images of the basis polynomials under T are $x(1) = x = 0(1) + 1(x) + 0(x^2)$ and $x(x) = x^2 = 0(1) + 0(x) + 1(x^2)$. The coordinates of these polynomials with respect to the standard basis therefore are $(0, 1, 0)$ and $(0, 0, 1)$. By putting them as columns we get the matrix representing T:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence rank equals 2 and the nullity equals $2 - 2 = 0$. $\square$

4. §4.3, # 2. [10 points] Are the matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}$$

linearly independent?

Solution. The easy way to solve this problem is to notice that

$$A_4 - A_3 = A_1.$$

Hence

$$-A_1 + 0 \cdot A_2 - A_3 + A_4 = 0,$$

which means that the matrices are not linearly independent. More difficult solution is to write down coordinates C_1, C_2, C_3, C_4 of these matrices with respect to the standard basis in the space of matrices (i.e. matrices E_{ij} which have 1 in the position ij and zero elsewhere) and then compute dimension of the span of the resulting vectors in $\mathbb{R}^4$. The 4×4 matrix C with the columns C_1, C_2, C_3, C_4 is

$$C = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 7 & 8 \end{bmatrix}.$$

Now, let's apply the Gauss-Jordan method to this matrix:

$$C \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 5 \\ 0 & 3 & 5 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence rank of C equals 3, therefore the span of the matrices $A_1, ..., A_4$ is 3-dimensional. Since we have 4 elements in a 3-dimensional space, they must be linearly dependent. □

5. §5.5, # 6. [10 points]

(a) Consider an $m \times n$ matrix P and $n \times m$ matrix Q. Show that

$$trace(PQ) = trace(QP).$$

Solution. Let a_{ij} and b_{ij} denote the entries of the matrices P and Q respectively. Then

$$trace(PQ) = \sum_{i,j} a_{ij} b_{ji},$$

where the sum is taken over all choices of indices i, j. Analogously,

$$trace(QP) = \sum_{i,j} b_{ij} a_{ji} = \sum_{i,j} a_{ji} b_{ij},$$

where the sum is taken over all choices of indices i, j. Therefore the summands appearing in $trace(PQ)$ are the same as the summands appearing in $trace(QP)$. Hence $trace(PQ) = trace(QP)$. □

(b) Compare the following inner products in $\mathbb{R}^{m \times n}$:

$$\langle A, B \rangle = trace(A^T B),$$

and

$$\langle\langle A, B \rangle\rangle = trace(AB^T).$$

Solution. Since for each square matrix S we have $trace(S) = trace(S^T)$, we get:

$$\langle A, B \rangle = trace(A^T B) = trace((A^T B)^T) = trace(B^T (A^T)^T) = trace(B^T A).$$

The latter equals to $trace(AB^T)$, according to part (a). Thus

$$\langle A, B\rangle = \langle\langle A, B\rangle\rangle. \quad \square$$

6. §5.5, # 9. [10 points] Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called even if $f(-t) = f(t)$ for all t. The function f is called odd if $f(-t) = -f(t)$ for all t.

Show that if f is an odd continuous function and g is an even continuous function then these functions are orthogonal as elements of the space $C[-1, 1]$, where the inner product is given by $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$.

Solution. Let's make the change of variables $t = -s$ in the integral:

$$\int_{-1}^{1} f(t)g(t)dt = \int_{t=-1}^{t=1} f(-s)g(-s)d(-s) =$$

$$= \int_{s=1}^{s=-1} (-1)^2 f(s)g(s)ds = -\int_{s=-1}^{s=1} f(s)g(s)ds.$$

(recall that reversing direction of the integration changes the sign of the integral). Hence

$$\int_{-1}^{1} f(t)g(t)dt = -\int_{-1}^{1} f(s)g(s)ds,$$

which means that the integral equals zero. Thus $\langle f, g\rangle = 0$ and therefore f is orthogonal to g. $\square$