

**MATHEMATICS 2270. Homework # 8: Solutions.**

Total: 50 points.

1. [5 points] Determine if the set

$$S = \{ \text{all continuous functions } f : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } \frac{1}{2}(f(x))^2 = \int_0^x f(t)dt \}$$

is a subspace of the space of all continuous functions? Hint: try to find a (nonzero) polynomial function which belongs to  $S$ .

Solution. First, note that the function  $f(x) = x$  belongs to the set  $S$ :

$$\int_0^x f(t)dt = \int_0^x tdt = x^2/2 = \frac{1}{2}(f(x))^2.$$

Next, for  $\alpha = 2$  we have:

$$\frac{1}{2}(\alpha f(x))^2 = 2x^2.$$

On the other hand,

$$\int_0^x \alpha f(t)dt = \int_0^x 2tdt = x^2.$$

Since the functions  $2x^2$  and  $x^2$  are different,  $\alpha f(x)$  does not belong to  $S$ . Therefore  $S$  is not a linear space.  $\square$

2. §4.1, # 3. [10 points] Is the following set a subspace in  $P_2$ . Find a basis if it is.

$$S = \{p(t) : p'(1) = p(2)\}.$$

Solution. First, let's show that  $S$  is a subspace.

(1) Zero function has zero derivative, hence it belongs to  $S$ .

(2) Suppose that  $p(t), q(t)$  belong to  $S$ , i.e.

$$p'(1) = p(2), q'(1) = q(2).$$

Then for their sum we get:

$$(p + q)'(1) = p'(1) + q'(1) = p(2) + q(2) = (p + q)(2).$$

Thus  $p + q$  belongs to  $S$ .

(3) Let  $p(t)$  be in  $S$  and  $\alpha$  be a scalar. Then

$$(\alpha p)'(1) = \alpha p'(1) = \alpha p(2) = (\alpha p)(2).$$

Thus  $\alpha p$  also belongs to  $S$ . Therefore  $S$  is a subspace.

Next, let's find a general solution of the equation  $p'(1) = p(2)$ ; let  $p(t) = a + bt + ct^2$ . Then  $p$  is in  $S$  if and only if

$$b + 2c = a + 2b + 4c,$$

i.e.

$$a = -b - 2c,$$

where  $b, c$  are parameters. The basis is given by first taking  $b = 1, c = 0$ , i.e.  $f(t) = -1 + t$ ; and then taking  $b = 0, c = 1$ , i.e.  $g(t) = -2 + t^2$ . Therefore a basis is

$$\{-1 + t, -2 + t^2\}. \quad \square$$

3. §4.1, # 26. [5 points] Find basis and determine the dimension of the following subspace in  $P_3$ :

$$S = \{f(1) = 0, \int_{-1}^1 f(t) dt = 0\}.$$

Solution. First, let's find a general formula for the polynomials in  $S$ . Let  $p(t) = a + bt + ct^2 + et^3$ . Then  $f(1) = a + b + c + e = 0$ . Furthermore,

$$0 = \int_{-1}^1 f(t) dt = \int_{-1}^1 (a + bt + ct^2 + et^3) dt = [at + bt^2/2 + ct^3/3 + et^4/4]_{-1}^1 = b + e/2.$$

Therefore we get:  $e = -2b, c = -a - b - e = -a - b + 2b = b - a$ . Thus the general polynomial in  $S$  is given by

$$a + bt + (b - a)t^2 - 2bt^3,$$

where  $a, b$  are arbitrary parameters. Hence to get a basis we first let  $a = 1, b = 0$  and then  $a = 0, b = 1$ , therefore a basis is

$$\{1 - t^2, t + t^2 - 2t^3\}.$$

The dimension equals 2. □

4. §4.1, # 28. [10 points] Find a basis and compute the dimension for the space of all  $2 \times 2$  matrices  $A$  which commute with

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution. We first find a general formula for the matrices  $A$  in  $S$ . Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then  $AB = BA$  means

$$\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}.$$

Therefore  $c = 0, d = a$  and we get:

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},$$

where  $a, b$  are arbitrary parameters. Whence to get a basis in  $S$  we first take  $a = 1, b = 0$  and then  $a = 0, b = 1$ . Thus a basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Dimension equals 2. □

5. §4.2, # 46. [10 points] Let  $X$  be the set of all students in our linear algebra class. Can you define operations on  $X$  which make  $X$  into a linear space? Explain.

Solution. The answer is that it is impossible to find such operations. Indeed, the number of students in our class is between 2 and 38 (actually, most of the time it is about 20, except for the days when we have a test). If one can define an operation of multiplication of the students by arbitrary real numbers then for each student  $x$  corresponding to a nonzero vector in  $X$  we would get:

$$\alpha \vec{x} \neq \beta \vec{x},$$

provided that  $\alpha \neq \beta$ . Since there are infinitely many real numbers  $\alpha$ , we would have infinitely many students in the class:

$$\vec{x}, 2\vec{x}, 3\vec{x}, 4\vec{x}, \dots,$$

which is absurd. □

6. [10 points] Use Cramer's Rule to find all values of the parameter  $a$  for which the system

$$\begin{cases} ax + y = 1 \\ 4x + ay = 2 \end{cases}$$

is consistent.

Solution. Consider the determinant of the matrix of the left hand side:  $a^2 - 4$ . If this determinant is different from zero, then the system always has a solution. Suppose that the determinant equals to zero, i.e.  $a = \pm 2$ . Then, by Cramer's Rule, if the system has a solution then:

$$\det(A_1) = \det(A_2) = 0,$$

$$\begin{vmatrix} 1 & 1 \\ 2 & a \end{vmatrix} = 0, \begin{vmatrix} a & 1 \\ 4 & 2 \end{vmatrix} = 0.$$

The last two equations are  $a - 2 = 0, 2a - 4 = 0$ . Thus  $a = 2$ . Finally, let's check if for  $a = 2$  the system has a solution. In this case the system is

$$\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases} \iff 2x + y = 1,$$

which has infinitely many solutions. Thus the system is consistent if and only if  $a \neq -2$ . □