MATHEMATICS 2270. Homework # 7: solutions.

Total=60 points.

1. [10 points] Compute the determinant using Laplace (row/column) expansion:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}.$$

Solution. Expaning by the first row we get:

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}.$$

The first determinant equals

$$\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -4 + (2-3) = -5.$$

The second determinant equals:

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore the entire determinant is $-5+1=-4$.

2. §6.2, # 4. [10 points] Compute the determinant using either Gauss-Jordan elimination or Laplace expansion:

$$\begin{vmatrix} 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 5 & 3 & 9 & 9 \\ 0 & 7 & 4 & 0 & 1 \\ 3 & 9 & 5 & 4 & 8 \end{vmatrix}.$$

Solution. We begin with the Laplace expansion oon the 1-st column, then expand by the second row:

$$3\begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 5 & 3 & 9 & 9 \\ 7 & 4 & 0 & 1 \end{vmatrix} = 3\left(2\begin{vmatrix} 2 & 0 & 1 \\ 5 & 9 & 9 \\ 7 & 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 & 0 \\ 5 & 3 & 9 \\ 7 & 4 & 0 \end{vmatrix}\right).$$

We now compute 3×3 determinants using expansion by the first row for the first one and by the third column for the second one:

$$\begin{vmatrix} 2 & 0 & 1 \\ 5 & 9 & 9 \\ 7 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 9 & 9 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 5 & 9 \\ 7 & 0 \end{vmatrix} = 18 - 63 = -45.$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 5 & 3 & 9 \\ 7 & 4 & 0 \end{vmatrix} = -9 \begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix} = -9(8-7) = -9.$$

Hence the entire determinant equals

$$3(2 \cdot (-45) + 2(-9)) = 3(-108) = -324. \quad \square$$

3. §6.2, # 12. [5 points] Consider 4×4 matrix A with the rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ such that $det(A) = 8$. Compute the determinant

$$\begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 + 9\vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix}.$$

Solution. By linearity of the determinant this determinant equals

$$\begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix} + 9 \begin{vmatrix} \vec{v}_1 \\ \vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix} = det(A) = 8,$$

since

$$\begin{vmatrix} \vec{v}_1 \\ \vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix} = 0$$

(it has twop equal rows). Therefore $det = 8$. $\square$

4. §6.2, # 27. [10 points] Consider a skew-symmetric $n \times n$ matrix A where n is odd, i.e. $A^T = -A$. Show that A is not invertible by verifying that $det(A) = 0$.

Solution. $det(A) = det(A^T) = det(-1 \cdot A) = (-1)^n det(A) = -det(A)$, since n is odd. Thus $det(A) = -det(A)$, which means that $det(A) = 0$. $\square$

5. §6.3, # 2. [5 points] Find area of the triangle defined by the vectors

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

Solution. The area of this triangle is half the area of the parallelogram defined by these vectors. The area of parallelogram equals

$$|det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}| = |6 - 56| = |-50| = 50.$$

Therefore the traiangle has area 25. $\square$

6. §6.3, # 14. [10 points] Find 3-volume of the parallelepiped defined by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution. Consider the matrix

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}.$$

Then the volume equals $\sqrt{|det(A^T A)|}$. We have:

$$det(A^T A) = det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{vmatrix} = 20 - 20 + 6 = 6.$$

Therefore the volume equals $\sqrt{106}$. □

7. [10 points] Use the "classical adjoint" to find inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution.

$$det(A) = 2 + 1 = 3.$$

The adjoint of A equals

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Therefore

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$