MATHEMATICS 2270. Homework # 6: Solutions.

Total: 80 points. If in Problems (2) and (3) students make mistakes in calculations, it is OK, the important thing is that they follow correct procedure.

1. [10 points] Determine if there exists an orthogonal transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

$$T\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Note that the magnitude of the vector

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

is the same as the magnitude of the vector

$$\vec{v_1'} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

i.e., $\sqrt{1+1+4} = \sqrt{6}$. The same applies to the vectors

$$\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{v_2}' = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

We next observe that $\vec{v_1} \cdot \vec{v_2} = 1 - 2 - 2 = -3$, which is different from $\vec{v_1}' \cdot \vec{v_2}' = 2 - 2 + 1 = 1$. Since orthogonal transformations preserve dot products, it follows that an orthogonal transformation T does not exist. $\square$

2. [20 points] Find matrix of the orthogonal projection to the span of the vectors:

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 5 \\ 3 \end{bmatrix}.$$

Solution. Consider the matrix $P = A(A^TA)^{-1}A^T$, this is the matrix of projection, where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 9 \\ -1 & 5 \\ 1 & 3 \end{bmatrix}.$$

To compute the matrix P we first compute the product A^TA:

$$A^TA = \begin{bmatrix} 4 & -10 \\ -10 & 116 \end{bmatrix}.$$

The inverse matrix equals

$$(A^T A)^{-1} = \begin{bmatrix} 29/91 & 5/182 \\ 5/182 & 1/91 \end{bmatrix}.$$

Hence

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 9 \\ -1 & 5 \\ 1 & 3 \end{bmatrix} (A^T A)^{-1} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 9 & 5 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 29/91 & 5/182 \\ 5/182 & 1/91 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 9 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 5/13 & 0 & -2/13 & 6/13 \\ 0 & 5/7 & 3/7 & 1/7 \\ -2/13 & 3/7 & 29/91 & -9/91 \\ 6/13 & 1/7 & -9/91 & 53/91 \end{bmatrix}.$$

3. §5.4, # 22. [20 points] Find the least square solution $\vec{x}_*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}.$$

Compute the error term $\epsilon = |A\vec{x}_* - \vec{b}|$.

Solution. The normal equation is $A^T A\vec{x} = A^T \vec{b}$. The augmented matrix of this system is

$$\left[\begin{array}{cc|c} 50 & 41 & 68 \\ 41 & 38 & 74 \end{array}\right].$$

After applying the Gauss-Jordan algorithm to this matrix we get:

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{-150}{73} \\ 0 & 1 & \frac{304}{73} \end{array}\right].$$

Therefore

$$\vec{x}_* = \begin{bmatrix} \frac{-150}{73} \\ \frac{304}{73} \end{bmatrix}.$$

To find the error term ϵ we compute the vector

$$A\vec{x}_* - \vec{b} = \frac{1}{73} \begin{bmatrix} -207 \\ 774 \end{bmatrix}.$$

The magnitide of this vector equals

$$\epsilon = \frac{45}{73}\sqrt{438}.$$

4. [10 points] §5.4, # 30. Fit a linear function of the form $f(t) = c_0 + c_1 t$ into the data points $(0,0), (0,1), (1,1)$. Make a sketch an explain why the solution makes sense.

Solution. The linear system which we are trying to solve using the least square method is

$$\begin{cases} f(0) = 0 \\ f(0) = 1 \\ f(1) = 1 \end{cases}$$

In other words,

$$\begin{cases} c_0 = 0 \\ c_0 = 1 \\ c_0 + c_1 = 1 \end{cases}$$

In matrix form: $A\overrightarrow{x} = \overrightarrow{b}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \overrightarrow{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

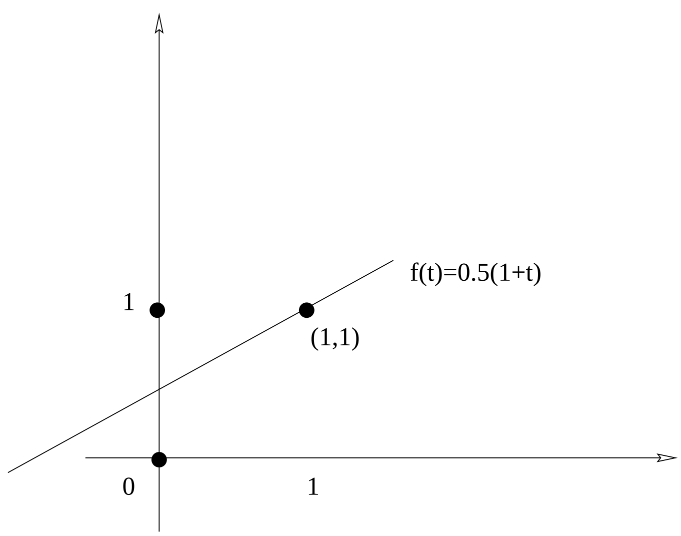
The normal equation $A^T A\overrightarrow{x} = A^T\overrightarrow{b}$ has the augmented matrix

$$C = \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 1 & 1 \end{array}\right].$$

Applying the Gauss-Jordan method to this matrix we get:

$$C_{rref} = \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right].$$

Therefore $c_0 = c_1 = 0.5$ and the function $f(t)$ equals $f(t) = 0.5(1 + t)$. Below is the sketch:



The solution makes sense since the line $f(t)$ passes between the points of the data.

5. §6.1, # 16. [10 points] Compute the determinant:

$$C = \begin{vmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 9 & 7 & 9 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{vmatrix}.$$

Solution. Let's try to find patterns without zeroes in this matrix. First, note that from the 4-th row we have to pick the entry 5, since everything else is zero. This means that the last column of the matrix is now taken. Next, look at the 2-nd row: the only nonzero entry we can take from it is 2 in the position $(2,4)$, since the other 2 is from the last column. Now, the last two columns in the matrix are taken. Next, observe that from the 1-st row we can take only 2 in the position $(1,3)$, since the other two numbers are from the columns which are already taken. Next, from the 2-nd column we can take only 9 in the position $(3,2)$. Lastly, from the first column we are forsed to use 3 in the position $(5,1)$.

The product of numbers in this pattern equals $3 \cdot 9 \cdot 2 \cdot 2 \cdot 5 = 540$. Finally, we have to decide if the number of inversions in this pattern is even or odd: we have 6 inversions (four involving the entry in the position $(5,1)$ and two involving the entry in the position $(2,3)$. Thus, since 6 is an even number,

$$det = +540. \quad \square$$

6. §6.1, # 32. [10 points] Find all values of λ such that the matrix

$$\lambda I_3 - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 8 & -2 \end{bmatrix}$$

is not invertible.

Solution. Consider the determinant of this matrix:

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & -8 & \lambda + 2 \end{vmatrix} = \lambda(\lambda^2 + 2\lambda - 8) = 0,$$

whenever the matrix is not invertible. Solving the equation $\lambda(\lambda^2 + 2\lambda - 8) = 0$ we get the solutions: $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -4$. Therefore the matrix is not invertible if and only if $\lambda = 0, \lambda = 2$, or $\lambda = -4$. $\square$