MATHEMATICS 2270. Homework # 5: Solutions.

Total: 50 points.

1. [10 points] Section 3.4, # 14. Find matrix B of the transformation

$$T(\vec{x}) = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix} \vec{x}$$

with respect to the basis

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solution. The transition matrix equals

$$S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Then

$$B = S^{-1} \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix} S.$$

We get:

$$S^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix} S = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 10 & 30 \end{bmatrix}.$$

Thus

$$B = S^{-1} \begin{bmatrix} 5 & -10 \\ 10 & 30 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & -10 \\ 10 & 30 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}.$$

Therefore

$$B = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}. \quad \square$$

2. [5 points] Are the following matrices similar:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad ?$$

Solution. Suppose that there is an invertible matrix

$$S = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

such that $SA = BS$. Then

$$\begin{bmatrix} x+y & y \\ z+w & w \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}.$$

It follows that $w=0$, hence $z=0$ as well since $z+w=0$. Moreover, since $x+y=x$, we get $y=0$. Thus the matrix S has the form

$$S = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

and zero determinant. Therefore S cannot be invertible. Hence A and B are not similar. $\square$

3. [5 points] Compute cosine of the angle between the following vectors and determine if the angle is acute or obtuse:

$$\vec{u} = (1,1,1), \quad \vec{v} = (2,3,1).$$

Solution. The dot product of these vectors equals $2+3+1=6$. The magnitudes of the vectors are

$$\sqrt{3}, \quad \sqrt{4+9+1} = \sqrt{14}.$$

Thus

$$\cos(\alpha) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{6}{\sqrt{14}}.$$

This number is positive, therefore the angle is acute. $\square$

4. [10 points] §5.1, # 23. Let V be a subspace in $\mathbb{R}^n$. Show that its orthogonal complement $V^\perp$ is also a subspace.

Solution. Let $\vec{v}_1, ..., \vec{v}_m$ be a basis in V. Then $V^\perp$ consists of all vectors $\vec{x}$ such that

$$\vec{v}_1 \cdot \vec{x} = 0, ..., \vec{v}_m \cdot \vec{x} = 0.$$

Now there are two ways to argue:

Argument 1. Note that the above equations amount to saying that $\vec{x}$ is in the kernel of the linear transformation given by the matrix

$$A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{bmatrix}.$$

Since the kernel of a linear transformation is always a subspace, $V^\perp$ is a subspace.

Argument 2, using definition of a subspace. Suppose that $\vec{x}, \vec{y}$ are vectors in $V^\perp$. This means that

$$\vec{v}_i \cdot \vec{x} = 0, \vec{v}_i \cdot \vec{y} = 0, \text{ for all } i = 1, ..., m.$$

Then

$$\vec{v}_i \cdot (\vec{x} + \vec{y}) = \vec{v}_i \cdot \vec{x} + \vec{v}_i \cdot \vec{y} = 0.$$

Hence $\vec{x} + \vec{y}$ is a vector in $V^\perp$.

Suppose that α is a real number, $\vec{x}$ is a vector in $V^\perp$. Then

$$(\alpha\vec{x}) \cdot \vec{v}_i = \alpha(\vec{x} \cdot \vec{v}_i) = 0, \text{ for all } i = 1, ..., m;$$

hence $\alpha\vec{x}$ is a vector in $V^{\perp}$. $\square$

5. [10 points] §5.1, # 26. Find projection of the vector $\vec{x} = (49, 49, 49)$ to the subspace spanned by the vectors $\vec{v_1} = (2, 3, 6)$ and $\vec{v_2} = (3, -6, 2)$.

Solution. First, note that

$$\vec{x} = 49 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Next, let's find an orthonormal basis $\vec{u}_1, \vec{u}_2$ in V. The dot product $\vec{v_1} \cdot \vec{v_2} = 6 - 18 + 12 = 0$, hence the vectors $\vec{v_1}, \vec{v_2}$ are already orthogonal. Their magnitudes are equal to $\sqrt{49} = 7$. Thus

$$\vec{u_1} = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{u_2} = \frac{1}{7} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Therefore

$$proj_V(\vec{x}) = (\vec{x} \cdot \vec{u_1})\vec{u_1} + (\vec{x} \cdot \vec{u_2})\vec{u_2}.$$

Since 49 cancels $7 \cdot 7$, the first summand equals

$$((1,1,1) \cdot (2,3,6)) \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = 11 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \\ 66 \end{bmatrix}.$$

Similarly, the second summand equals

$$((1,1,1) \cdot (3,-6,2)) \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}.$$

Therefore

$$proj_V(\vec{x}) = \begin{bmatrix} 22 \\ 33 \\ 66 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}. \quad \square$$

6. [10 points] §5.2, # 10. Using Gram-Schmidt process orthogonalize the basis $\vec{v_1} = (1, 1, 1, 1), \vec{v_2} = (6, 4, 6, 4)$.

Solution. $|\vec{v_1}| = \sqrt{4} = 2$. Thus

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let L be the line through the vector $\vec{u_1}$. Then

$$proj_L(\vec{v_2}) = (\vec{v_2} \cdot \vec{u}_1)(\frac{1}{2}) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{20}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}.$$

Hence

$$\overrightarrow{w_2} := proj_{L^\perp}(\overrightarrow{v_2}) = \overrightarrow{v_2} - proj_L(\overrightarrow{v_2}) =$$

$$\begin{bmatrix} 6 \\ 4 \\ 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Finally, let's normalize the vector $\overrightarrow{w_2}$:

$$\overrightarrow{u_2} = \overrightarrow{w_2}/|\overrightarrow{w_2}| = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore the orthonormal basis equals

$$\overrightarrow{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \overrightarrow{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}. \quad \square$$