MATHEMATICS 2270. Solutions for Homework # 3.

For each matrix A in the problems (1) and (2) find vectors which span the kernel of A:

1. [10 points]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

Solution. Let's compute the reduced row eachelon form of the matrix A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the solutions of the equation $A\vec{x} = \vec{0}$ are of the form:

$$\vec{x} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Thus the kernel is spanned by the vector $(-1, -1, 1)$. □

2. [10 points]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}.$$

Solution. Let's compute the reduced row eachelon form of the matrix A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}.$$

Thus $x = -3x_3$, $x_2 = x_3$, x_3 is any real number. Hence the solutions of the equation $A\vec{x} = \vec{0}$ are of the form:

$$\vec{x} = \begin{bmatrix} -3t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Thus the kernel is spanned by the vector $(-3, 1, 1)$. □

For each matrix A in the problems (3) and (4) find vectors which span the image of A. Give as few vectors as possible:

3. [5 points]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

Solution. First, note that the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

span the image of A. However this collection of vectors is not minimal. One way to solve the problem therefore is to note that the vector $\overrightarrow{w}$ equals $\overrightarrow{u}+\overrightarrow{v}$. Hence the vectors $\overrightarrow{u}, \overrightarrow{v}$ also span the image of A. Since the vectors $\overrightarrow{u}, \overrightarrow{v}$ are not proportional to each other, it is impossible to find the smaller set of spanning vectors.

Here is another solution, using the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \Rightarrow \ldots \Rightarrow A_{rref} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

see Problem 1. In the matrix A_{rref} only the first two columns contain leading entries. Therefore the first two columns of A will span the image of A. The first two columns of A are the vectors $\overrightarrow{u}, \overrightarrow{v}$. Hence the set $\{\overrightarrow{u}, \overrightarrow{v}\}$ is the smallest spanning set of vectors for the image of A.

4. [5 points]

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Solution. Note that the column vectors

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \overrightarrow{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \overrightarrow{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

span the image of A. Note that $\overrightarrow{v} = 2\overrightarrow{u} - \overrightarrow{w}$. Hence the vectors $\overrightarrow{u}, \overrightarrow{w}$ also span the image of A. Since these vectors are not proportional to each other, there is no smaller set of spanning vectors. Thus the set of vectors $\{\overrightarrow{u}, \overrightarrow{w}\}$ is the smallest spanning set for the image of A.

Here is another solution using the reduced row echelon form of A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \Rightarrow \ldots \Rightarrow A_{rref} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix},$$

see Problem 2. The matrix A_{rref} has only 1-st and 2-nd column containing leading entries. Therefore the first and second columns of A form the smallest spanning set for the image of A. Thus the set of vectors $\{\overrightarrow{u}, \overrightarrow{v}\}$ is the smallest spanning set for the image of A. Note that this answer is different from the answer obtained before. This is OK however. One can check that any set of two non-proportional (nonzero) vectors in $\mathbb{R}^2$ spans the image of A.

5. §2.3, # 10. [10 points] . Decide if the matrix A is invertible and if it is, find the inverse.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution.

$$\tilde{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & -5 & 2 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right].$$

Therefore

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

6. [10 points]. Using row echelon form reduction find the inverse matrix A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.

$$\tilde{A} = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus the matrix

$$B = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is the inverse to the matrix A. □