MATHEMATICS 2270. Homework # 2: Solutions.

1. §2.1, # 6. [5 points] Decide if the transformation is linear and find its matrix if it is the case:

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution.

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix}.$$

This transformation is linear and its matrix is

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}. \quad \square$$

2. §2.1, # 20. [10 points] Give geometric interpretation to the following linear transformation. Determine if it is invertible, find the inverse if it exists.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solution. The transformation is given by the formula:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

This means that T simply interchanges the coordinates in the xy-plane. Thus T is the reflection in the line $x = y$. The transformation is clearly invertible, its inverse is T itself. $\square$

3. §2.2, # 24 (a, b). [10 points] Consider the linear transformation

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Sketch the image of the unit square under T.

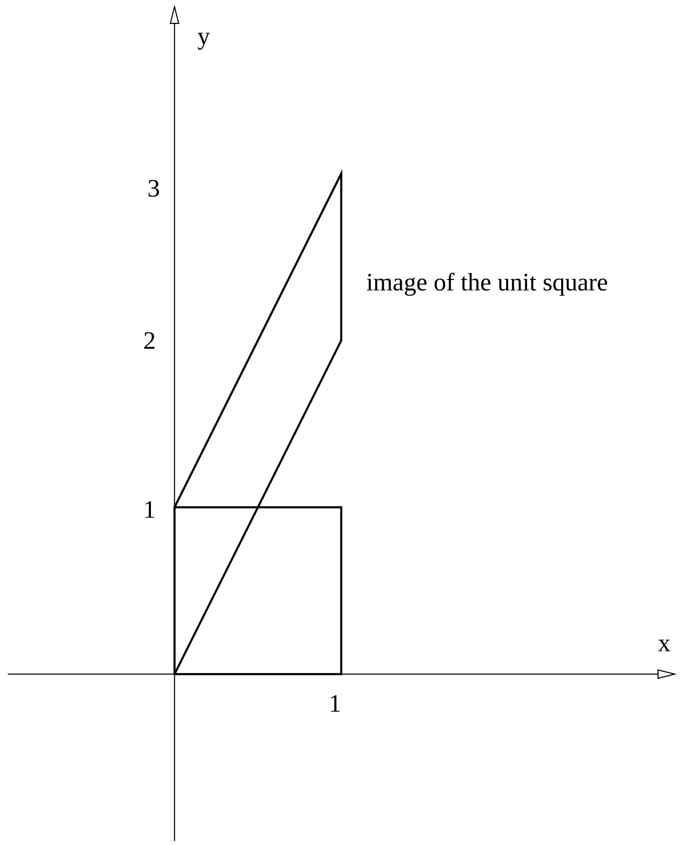
(b) Show that T is a shear.

Solution. (b) Note that

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x + y \end{bmatrix}.$$

Thus each point on the y-axis maps by T into itself. Now, fix the x-coordinate to be a constant c and consider the line $x = c$ parallel to the y-axis. The transformation T does not change the x-coordinate, and each point on the line $x = c$ gets shifted vertically by $2x$ amount. Therefore T is a shear along the y-axis.

(a) The image of the unit square is shown below:



4. §2.1, # 26. [10 points] Find the inverse of the following nonlinear transformation if it exists:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^3 + x_2 \end{bmatrix}.$$

Solution. Given vector $\vec{y}$ let's try to find vectors $\vec{x}$ which map to $\vec{y}$, i.e. let's try to solve (for x_1, x_2) the equation

$$T\vec{x} = \vec{y}\ .$$

We get:

$$\begin{cases} y_1 = x_2 \\ y_2 = x_1^3 + x_2 \end{cases}$$

Substituting the value $x_2 = y_1$ into the second equation we get

$$x_1^3 = y_2 - y_1,$$

$$x_1 = (y_2 - y_1)^{1/3}.$$

Since the cubic root is always defined and is unique, we find that the inverse transformation is given by the formula:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (y_2 - y_1)^{1/3} \\ y_1 \end{bmatrix}. \quad \square$$

5. §2.3, # 38. [5 points] Find the inverse matrix to

$$A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}.$$

Solution. The determinant of A equals $(1)(-1) - (k)(0) = -1$. Thus the inverse matrix is:

$$A^{-1} = (-1)\begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix} = A^{-1} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} = A.$$

6. [10 points] Compute the product of matrices:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}.$$

Thus the triple product equals

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [a \cdot 0 + b \cdot 1 + c \cdot 0] = [b]. \quad \square$$