MATHEMATICS 2270. Homework # 12: Not to be handed in!.

Solutions.

1. §8.1, # 8. Find diagonal matrix D and orthogonal matrix S so that $S^{-1}AS = D$.

$$A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}.$$

Solution. The eigenvalues of A are the roots of the polynomial

$$(\lambda - 3)(\lambda + 5) - 9 = \lambda^2 + 2\lambda - 24.$$

The roots are $\lambda = -1 \pm \sqrt{25}$, $\lambda_1 = 4, \lambda_2 = -6$.

The basis of E_4 is the basis in the kernel of the matrix

$$4I - A = \begin{bmatrix} 4-3 & -3 \\ -3 & 4+5 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

Hence the basis is $(3,1)$. The basis of E_{-6} is the basis in the kernel of the matrix

$$-6I - A = \begin{bmatrix} -6-3 & -3 \\ -3 & -6+5 \end{bmatrix} = \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence the basis is $(1,-3)$. Note that the vectors $(3,1)$ and $(1,-3)$ are already orthogonal. Hence we just normalize them:

$$\vec{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Therefore

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}, S = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}.$$

2. §8.1, # 10. Find diagonal matrix D and orthogonal matrix S so that $S^{-1}AS = D$.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

Solution. The eigenvalues are $\lambda_1 = 9$ (of multiplicity 1) and 0 of multiplicity 2. The basis for E_9 is the vector $(1,-2,2)$. Normalizing this vector we get

$$\vec{u}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

An eigenbasis for E_0 is $v_2 = (-2,0,1), v_3 = (2,1,0)$. Note that these vectors are not orthogonal to each other. Let's apply Gramm-Schmidt:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

$$Proj_{u_2}(v_3) = \frac{-4}{5} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$w_3 = v_3 - Proj_{u_2}(v_3) = \begin{bmatrix} 2/5 \\ 1 \\ 4/5 \end{bmatrix}.$$

Finally, by normalizing this vector we get:

$$\vec{u}_3 = \frac{5}{\sqrt{21}} \begin{bmatrix} 2/5 \\ 1 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{21} \\ 5/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$$

Thus

$$D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 1/3 & -2/\sqrt{5} & 2/\sqrt{21} \\ -2/3 & 0 & 5/\sqrt{21} \\ 4/\sqrt{21} & & \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{21} \end{bmatrix}. \quad \square$$

3. §8.2, # 4. Determined definiteness of the quadratic form $q(x, y) = 6x^2 + 4xy + 3y^2$.

Solution. The matrix of this form is

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}.$$

It has determinant $18 - 4 = 14 > 0$ and trace $9 > 0$. Hence the form is positive definite.

4. §8.2, # 7. Determined definiteness of the quadratic form $q(x, y) = 3x_2^2 + 4x_1x_3$.
Solution. The matrix of this form is

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

The determinant of this matrix is $-12 < 0$, hence A is indefinite.

5. §8.2, # 1. For the quadratic form q find the matrix A. $q(x, y) = 6x^2 - 7xy + 8y^2$.
Solution.

$$A = \begin{bmatrix} 6 & 7/2 \\ 7/2 & 8 \end{bmatrix}.$$