MATHEMATICS 2270. Solutions for homework # 11.

Total: 65 points.

1. §7.4, # 10. [5 points] Diagonalize if possible the matrix

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

Solution. The characteristic polynomial equals

$$\begin{vmatrix} \lambda - 3 & -4 \\ 1 & \lambda + 1 \end{vmatrix} = \lambda^2 - 2\lambda - 3 + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

Hence we get single eigenvalue $\lambda = 1$ of the algebraic multiplicity 2. Let's find the eigenvectors:

$$\lambda I - A = \begin{bmatrix} 1-3 & -4 \\ 1 & 1+1 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

has rank 1, which means that the geometric multiplicity (i.e. the dimension of E_1) equals 1. Thus geometric multiplicity is strictly less than the algebraic multiplicity, which implies that A is not diagonalizable. □

2. §7.4, # 20. [10 points] Diagonalize if possible the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution. The characteristic polynomial equals

$$\begin{vmatrix} \lambda - 1 & 0 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3 - (\lambda - 1) =$$

$$= (\lambda - 1)[(\lambda - 1)^2 - 1] = (\lambda - 1)(\lambda - 1 + 1)(\lambda - 1 - 1) = (\lambda - 1)(\lambda - 2)\lambda.$$

Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 0$ each having multiplicity 1. Therefore A is diagonalizable. Now, let's find the eigenbasis. For λ_1 we have to find a basis in the kernel of

$$I - A = \begin{bmatrix} 1-1 & 0 & -1 \\ -1 & 1-1 & -1 \\ -1 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvectors (x, y, z) satisfy $z = 0, -x - z = 0, x = 0$, y is any number. Thus the eigenvector is $\vec{v}_1 = (0, 1, 0)$. For λ_2 we have to find a basis in the kernel of

$$2I - A = \begin{bmatrix} 2-1 & 0 & -1 \\ -1 & 2-1 & -1 \\ -1 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvectors (x, y, z) satisfy $x = z, y = 2z$, z is any number. Thus the eigenvector is $\vec{v}_2 = (1, 2, 1)$. For λ_3 we have to find a basis in the kernel of

$$-A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvectors (x, y, z) satisfy $x = -z, y = 0$, z is any number. Thus the eigenvector is $\vec{v}_2 = (-1, 0, 1)$.

Therefore the matrices D and S are:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \quad \square$$

3. §7.4, # 32. [15 points] For the matrix A compute A^t, where t is a natural number. Find the vector $A^t\vec{v}$, where $\vec{v} = (1, 2)$.

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}.$$

Solution. Let's first diagonalize the matrix A. The characteristic polynomial is

$$\begin{vmatrix} \lambda - 4 & 2 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 5\lambda + 6.$$

The eigenvalues are: $\lambda_1 = 2, \lambda_2 = 3$. The eigenspace for $\lambda_1 = 2$ is the kernel of

$$\begin{bmatrix} 2-4 & 2 \\ -1 & 2-1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence $\vec{v}_1 = (1, 1)$. The eigenspace for $\lambda_1 = 2$ is the kernel of

$$\begin{bmatrix} 3-4 & 2 \\ -1 & 3-1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

Hence $\vec{v}_2 = (2, 1)$. Thus

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The inverse of S is the matrix

$$S^{-1} = -\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

We have $A = SDS^{-1}$ and $A^t = SD^tS^{-1}$:

$$A^t = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2^t + 2 \cdot 3^t & 2^{t+1} - 2 \cdot 3^t \\ -2^t + 3^t & 2^{t+1} - 3^t \end{bmatrix}.$$

Finally,

$$A^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2^t + 2\cdot 3^t + 4\cdot 2^t - 4\cdot 3^t \\ -2^t + 3^t + 4\cdot 2^t - 2\cdot 3^t \end{bmatrix} = \begin{bmatrix} 3\cdot 2^t - 2\cdot 3^t \\ 3\cdot 2^t - 3^t \end{bmatrix}. \quad \square$$

4. §7.4, # 50. [10 points] Do not forget to find all eigenvalues and bases of eigenvectors for the corresponding eigenspaces in this problem!

Find all eigenvalues and eigenvectors of the linear transformation $T : P_2 \to P_2$, $T(f) = f(x-3)$. Here P_2 is the space of all polynomials of degree ≤ 2.

Solution. The equation for the eigenvalues/eigenvectors is:

$$T(f) = \lambda f, f(x-3) = \lambda f(x).$$

The matrix of T (with respect to the standard basis) is:

$$A = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

The only eigenvalue is $\lambda = 1$. To find the eigenvectors consider the kernel of

$$\begin{bmatrix} 0 & 3 & -9 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the basis of eigenvectors in E_1 consists of the single vector $(1,0,0)$, which corresponds to the polynomial $p(x) = 1$. In particular, there is no eigenbasis and thus T is not diagonalizable. $\square$

5. [20 points] Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Find all complex eigenvalues of A and construct a complex eigenbasis. Use diagonalization to compute the powers A^n for all natural numbers n. Hint: represent n as $n = 4k + p$ where $0 \leq p < 4$. Check that the power A^n depends on the number p and not on k.

Solution. First, let's find the complex eigenvalues of A; by expanding the determinant along the first row we get:

$$0 = \begin{vmatrix} \lambda & 0 & 0 & -1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{vmatrix} + \begin{vmatrix} -1 & \lambda & 0 \\ 0 & -1 & \lambda \\ 0 & 0 & -1 \end{vmatrix} = \lambda^4 - 1.$$

Hence $\lambda^4 = 1$, which has the solutions $\pm 1, \pm i$, all have multiplicity 1. This means that the matrix is diagonalizable and the diagonal matrix D is

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Now, let's find the complex eigenbasis. For $\lambda = 1$ the eigenvector $\vec{v_1}$ spans the kernel of the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The reduced echelon form of this matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the first eigenvector is $\vec{v_1} = (1, 1, 1, 1)$.

For $\lambda = -1$ the eigenvector $\vec{v_2}$ spans the kernel of the matrix:

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

The reduced echelon form of this matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the second eigenvector is $\vec{v_2} = (-1, 1, -1, 1)$.

For $\lambda = i$ the eigenvector $\vec{v_3}$ spans the kernel of the matrix:

$$\begin{bmatrix} i & 0 & 0 & -1 \\ -1 & i & 0 & 0 \\ 0 & -1 & i & 0 \\ 0 & 0 & -1 & i \end{bmatrix}.$$

The reduced echelon form of this matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the third eigenvector is $\vec{v_3} = (-i, -1, i, 1)$.

For the eigenvalue $-i$ which is complex conjugate to i, the eigenvector is the complex conjugate of $\vec{v_3}$, i.e. $\vec{v_4} = (i, -1, -i, 1)$.

Thus the matrix S equals

$$S = \begin{bmatrix} 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now let's compute the powers of the matrix A: $A^n = SD^nS^{-1}$. Note that D has on its diagonal $\pm 1, \pm i$; since each of these numbers to the 4-th power equals 1, the powers of D (and hence the powers of A) depend on the number p and not on k; where $n = 4k + p$.

If $n = 4k$, then $p = 0$, and hence $D^p = D^n = I$; thus $A^n = SS^{-1} = I$.

If $n = 4k+1$, then $p = 1$, and hence $D^p = D^n = D$; thus $A^n = SDS^{-1} = A$.

If $n = 4k+2$, then $p = 2$, and hence $A^n = A^2$:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Finally, if $n = 4k+3$ then $A^n = A^3$ equals

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

6. §7.5, # 22. [5 points] Find all complex eigenvalues of the matrix A:

$$\begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}.$$

Solution. The characteristic polynomial equals

$$\begin{vmatrix} \lambda - 1 & -3 \\ 4 & \lambda - 10 \end{vmatrix} = (\lambda - 1)(\lambda - 10) + 12 = \lambda^2 - 11\lambda + 10 + 12 = \lambda^2 - 11\lambda + 22.$$

The roots of this polynomial are:

$$\lambda_{1,2} = \frac{1}{2}[11 \pm \sqrt{121 - 88}] = \frac{1}{2}[11 \pm \sqrt{33}].$$

Hence the eigenvalues are $\frac{1}{2}[11 + \sqrt{33}], \frac{1}{2}[11 - \sqrt{33}]$. □

The original homework had a typo: §7.4, # 22. Give full credit (5 points) for correct solution of the problem:

For which constants a and b is the matrix A diagonalizable:

$$A = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}.$$

Solution. The characteristic polynomial of this matrix is $(\lambda - 1)(\lambda - b)$. This polynomial has roots 1 and b. They have multiplicity 1 if $1 \neq b$. Thus if $1 \neq b$, a arbitrary, then A is diagonalizable. Suppose that $b = 1$. Then the eigenspace E_1 is the kernel of

$$I - A = \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}.$$

The latter has nullity 1 if $a \neq 0$, hence for $b = 1, a \neq 0$ the matrix A is not diagonalizable. If $a = 0$ then A is already diagonal.

Answer: A is diagonalizable unless $b = 1, a \neq 0$. □