MATHEMATICS 2270. Homework # 10: Solutions.

Total= 40 points.

1. §7.1, # 6. [5 points] If $\vec{v}$ is an eigenvector of both A and B, is $\vec{v}$ necessarily an eigenvector of AB?

Solution. We have $A\vec{v} = \lambda\vec{v}$, $B\vec{v} = \mu\vec{v}$, hence

$$A(B\vec{v}) = A\mu\vec{v} = \mu A\vec{v} = (\mu\lambda)\vec{v}.$$

Hence $\vec{v}$ is an eigenvector of AB. □

2. §7.1, # 34. [10 points] Find a 2×2 matrix A such that $\vec{v} = (3, 1)$ and $\vec{u} = (1, 2)$ are its eigenvectors with eigenvalues 5 and 10 respectively. [Hint: first write down the matrix of the corresponding linear transformation using the coordinates associated with the basis of eigenvectors and then do the change of coordinates.]

Solution. Let $T(\vec{x}) = A\vec{x}$. We start with the basis $\mathcal{B} = \{(3, 1), (1, 2)\}$, with respect to this basis the transformation T is given by the matrix:

$$B = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix},$$

since $T(\vec{v}) = 5\vec{v}, T(\vec{u}) = 10\vec{u}$. The transition matrix from $\mathcal{B}$-coordinates to the standard coordinates is

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

whose inverse is

$$S^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Therefore

$$A = SBS^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}.$$

3. §7.2, # 8. [5 points] Use characteristic polynomial to find eigenvalues of the matrix A and their algebraic multiplicities:

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}.$$

Solution. The characteristic polynomial equals

$$f_A(\lambda) = \begin{vmatrix} \lambda+1 & 1 & 1 \\ 1 & \lambda+1 & 1 \\ 1 & 1 & \lambda+1 \end{vmatrix} = (\lambda+1)^3 + 2 - 3(\lambda+1) = \lambda^3 + 3\lambda^2 = \lambda^2(\lambda+3).$$

Therefore $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2 and $\lambda = -3$ is an eigenvalue of algebraic multiplicity 1. □

4. §7.2, # 12. [10 points] Use characteristic polynomial to find eigenvalues of the matrix A and their algebraic multiplicities:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}.$$

Solution. The characteristic polynomial equals

$$f_A(\lambda) = \begin{vmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \lambda+1 & 0 & 0 \\ 0 & 0 & \lambda-3 & 4 \\ 0 & 0 & -2 & \lambda+3 \end{vmatrix} =$$

$$[(\lambda-2)(\lambda+1)-1][(\lambda-3)(\lambda+3)+8] = [\lambda^2-\lambda-3][\lambda^2-1].$$

Thus $1, -1, \frac{1}{2}(1+\sqrt{13})$ and $\frac{1}{2}(1-\sqrt{13})$ are eigenvalues of algebraic multiplicity 1. □

5. §7.3, # 12. [5 points] Find all real eigenvalues of A, find basis of each eigenspace and if possible find an eigenbasis.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. The characteristic polynomial equals $(\lambda-1)^3$, since the matrix is upper triangular. Thus $\lambda = 1$ is the only eigenvalue. To find the corresponding eigenspace E_1 consider the equation

$$(I-A)\overrightarrow{x} = \overrightarrow{0},$$

which has the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix},$$

where x_1 is a parameter. Hence the basis in E_1 consists of the single vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

In particular, there is no eigenbasis. □

6. §7.3, # 36. [5 points] Are the following matrices similar:

$$A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}?$$

Solution. Traces of similar matrices should be the same. However $tr(A) = 3$, $tr(B) = 4$ are different. Hence A is not similar to B. (Note that we cannot use the determinants: both matrices have the same determinant -5.)