MATHEMATICS 2270. Solutions of Homework # 1.

1. [10 points] Write the augmented matrix and find (using Gauss-Jordan elimination algorithm for matrices) all solutions of the linear system:

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 3 \\ 2x_1 + x_2 + 3x_3 + 2x_4 = 5 \\ 3x_1 + 2x_2 + 4x_3 = 8 \end{cases}$$

Solution. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 3 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 4 & 0 & 8 \end{array}\right].$$

Gauss-Jordan elimination yields:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 3 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 4 & 0 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & -1 & -4 & 1 \\ 0 & -1 & 1 & 6 & -1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & -1 & -4 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The last is a matrix A_{rref} in the RREF. Hence we declare the only nonleading unknown x_3 to be the parameter t. From the three equations given by A_{rref} we find:

$$x_1 = 2 - 2t, \quad x_2 = 1 + t, \quad x_4 = 0.$$

Hence the set of solutions of the linear system is

$$\{x_1 = 2 - 2t, x_2 = 1 + t, x_3 = t, x_4 = 0; t \text{ is any number.}\} \quad \square$$

2. Problem 14, section 1.1. [10 points] Find all solutions of the linear system. Describe your solution in terms of intersecting planes. Do not need to sketch the planes.

$$\begin{cases} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{cases}$$

Solution. Using Gauss elimination we find:

$$\begin{cases} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{cases} \Rightarrow \begin{cases} x + 4y + z = 0 \\ -3y + 3z = 0 \\ -6y + 6z = 1 \end{cases} \Rightarrow \begin{cases} x + 4y + z = 0 \\ y - z = 0 \\ 0 = 1 \end{cases}$$

Thus the system has no solutions. Geometrically this means that we have three planes in the 3-space which intersect pairwise but they have no points of triple intersection. $\square$

3. Problem 18, section 1.2. [5 points] Determine which of the matrices below are in reduced row echelon form:

$$(a) \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (b) \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(c) \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}. \quad (d) \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

Solution. (a) and (c) are not in RREF, (b) and (d) are in RREF.

4. Problem 9, section 1.2. [10 points] Find all solutions of the linear system:

$$\begin{cases} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{cases}$$

Solution. Using augmented matrices we get:

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array}\right].$$

The last is a matrix in RREF, the parameters are $x_2 = t, x_5 = s, x_6 = u$. Hence the set of solutions is:

$$x_1 = -2t - s + u, \quad x_2 = t, \quad x_3 = 1 + s - u,$$

$$x_4 = -2s + u + 2, \quad x_5 = s, \quad x_6 = u,$$

where t, s, u are arbitrary numbers. □

5. Problem 2, section 1.3. [5 points] Find rank of the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. One way to solve this problem is using the material from the class: this is 3 by 3 upper triangular matrix with no zeroes on the diagonal. Hence its rank equals 3. Here is another solution, based on the Gauss-Jordan method:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The last is a matrix in RREF which has 3 nonzero rows. Thus rank equals 3. □

6. Problem 3, section 1.3. [5 points] Find rank of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution. We apply Gauss-Jordan algorithm to this matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last is a matrix in RREF which has one nonzero row. Hence rank equal 1. □

7. Problem 18, section 1.3. [5 points] Compute product:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution. Product equals:

$$\begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 \\ 3 \cdot 1 + 4 \cdot 2 \\ 5 \cdot 1 + 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}.$$