Math 3210-2. Final Test (Sample): Solutions.

1. State (write down to the best of your abilities) the Completeness Axiom for the real numbers.

Solution. See the textbook.

2. (a) Prove that if $(x_n), (y_n)$ are convergent sequences of real numbers then

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

(b) Give an example of the sequences $(x_n), (y_n)$ of real numbers such that neither (x_n), nor (y_n) converges, but the sequence $(x_n + y_n)$ converges.

Solution. (a) See the textbook for the proof.

(b) Take for instance, $x_n = (-1)^n, y_n = -(-1)^n$. Then $x_n + y_n = 0$ gives a convergent sequence. However neither (x_n), nor (y_n) converge since they contain subsequences which converge to different limits: $(1), (-1)$. $\square$

3. For the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

compute the image $f(E)$ of the set $E = [-2, 3]$. As in other problems you can use the fact that polynomial functions are continuous on $\mathbb{R}$.

Solution. Let $E_+ = E \cap [0, \infty) = [0, 3]$, $E_- = E \cap (-\infty, 0) = [-2, 0)$. Then $E = E_+ \cup E_-$. Thus $f(E) = f(E_+) \cup f(E_-)$. Note that the restriction of f to E_+ is given by the formula $f(x) = x^2$. Thus the function f is strictly increasing on $[0, 3]$ (since f is continuous on E_+ and its derivative $2x$ is positive on $(0, 3)$). Hence by the monotone function theorem, $f([0, 3]) = [0, 9]$.

Next note that the restriction of f to $E_- = [-2, 0)$ is given by the formula $f(x) = -x^2$. The derivative of this function is $-2x$ is positive on $[-2, 0)$. The function f is also continuous on $[-2, 0)$. Thus f is strictly increasing on $[-2, 0)$ and by the monotone function theorem we have $f(E_-) = [-4, 0)$. Thus $f(E) = [-4, 0) \cup [0, 9] = [-4, 9]$. $\square$

4. Compute the limit (or show that it does not exist)

$$\lim_{x\to 0} x \cos(\frac{1}{x}).$$

Solution. The function x is continuous, hence $\lim_{x\to 0} x$; the function $\cos(x)$ is bounded. Hence by the theorem on product of a function which has 0 limit and a bounded function we have:

$$\lim_{x\to 0} x \cos(\frac{1}{x}) = 0. \quad \square$$

5. Show that for each real number $a \in [0, 1)$,

$$\lim_{n\to\infty} a^n = 0.$$

Solution. See the textbook.

6. Using the definition of the limit compute

$$\lim_{x\to 0} \frac{1}{x^2}.$$

(Here you can first "guess" what the limit is, but then you would have to give a proof that your answer is correct.)

Solution. The limit equals $+\infty$. To prove this, for each $C \in \mathbb{R}$ we have to find $\delta > 0$ such that if $0 < |x| < \delta$ then $1/x^2 > C$. If $C \leq 0$ then any δ will work, so in this case take $\delta = 1$. Now consider the case $C > 0$. Then the inequality $1/x^2 > C$ is equivalent to $1/|x| > \sqrt{C}$ (since x^2 is strictly increasing on $[0, \infty)$. The inequality $1/|x| > \sqrt{C}$ is equivalent to $|x| < 1/\sqrt{C}$. Hence we can take $\delta = 1/\sqrt{C}$. $\square$

7. Prove the following inequality for all $x \geq 1$:

$$e^{x-1} > \log(x).$$

Solution. The function $f(x) = e^{x-1} - \log(x)$ is differentiable on $[1, \infty)$. Let's check that $f(x)$ is increasing on $[1, \infty)$. The derivative of f equals $f'(x) = e^{x-1} - 1/x$. The derivative of e^{x-1} is positive. The derivative of $-1/x$ equals $1/x^2 > 0$, hence the function $f'(x)$ has positive derivative on $[1, \infty)$, thus the function $f'(x)$ is increasing on $[1, \infty)$ and $f'(x) \geq f'(1) = 1 - 1 = 0$ for each $x \geq 1$. Thus $f'(x) \geq 0$ for $x \geq 1$ which implies that f is increasing on $[1, \infty)$ which implies that $f(x) \geq f(1) = 1 > 0$ for each $x \geq 1$. Thus $e^{x-1} > \log(x)$ for each $x \geq 1$. $\square$

8. Prove that Dirichlet's function $f(x) = 0, x \notin \mathbb{Q}$, $f(x) = 1, x \in \mathbb{Q}$, is not integrable on $[0, 1]$.

Solution. Let $P = \{0 = t_0 < t_1 < ... < t_n = 1\}$ be a partition of $[0, 1]$. Then

$$U(f, P) = \sum_{k=1}^{n} \sup(f|_{[t_{k-1}, t_k]})(t_k - t_{k-1}).$$

By density of rational numbers, each interval $[t_{k-1}, t_k]$ contains a rational number, hence $\sup(f|_{[t_{k-1}, t_k]}) = 1$.

Similarly,

$$L(f, P) = \sum_{k=1}^{n} \inf(f|_{[t_{k-1}, t_k]})(t_k - t_{k-1}),$$

$\inf(f|_{[t_{k-1}, t_k]}) = 0$ since irrational numebrs are dense too. Hence

$$L(f, P) = \sum_{k=1}^{n} 0(t_k - t_{k-1}) = 0,$$

$$U(f,P) = \sum_{k=1}^{n}(t_k - t_{k-1}) = 1.$$

Hehce $U(f) = \inf_P U(f,P) = 1$, $L(f) = \sup_P L(f,P) = 0$. Thus $U(f) \neq L(f)$ and so the function is not integrable. $\square$

9. State (write down to the best of your abilities) the Bolzano-Weierstrass Theorem.

Solution. See the textbook.

10. Compute the derivative $f'(0)$ for the function

$$f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0 \\ \cos(x) & \text{if } x < 0 \end{cases}$$

or show that this derivative does not exist.

Solution. $f(0) = 0 + 1 = 1$. $f'(0) = \lim_{x\to 0} \frac{f(x)-f(1)}{x}$. To compute this limit frist consider the limit from the right:

$$\lim_{x\to 0+} \frac{f(x)-f(1)}{x} = \lim_{x\to 0+} \frac{x^2+1-1}{x} = \lim_{x\to 0+} x = 0.$$

Now consider the limit from the left:

$$\lim_{x\to 0-} \frac{f(x)-f(1)}{x} = \lim_{x\to 0-} \frac{\cos(x)-1}{x} = (\cos)'(0) = -\sin(0) = 0.$$

Thus the limits from the left and right are equal to zero, which implies that $f'(0) = \lim_{x\to 0} \frac{f(x)-f(1)}{x} = 0$. $\square$